Magnetic Laplacian
in sharp three-dimensional cones

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Schrödinger operator

Notation

Ω	simply connected domain of $\mathbb{R}^2$ or $\mathbb{R}^3$
$\mathbf{B}$	non-vanishing magnetic field
$\mathbf{A}$	magnetic potential satisfying $\operatorname{curl} \mathbf{A} = \mathbf{B}$
$h > 0$	semi-classical parameter

Semiclassical Magnetic Laplacian

$$H_h(\mathbf{A}, \Omega) := (-ih\nabla + \mathbf{A})^2 \quad \text{on } \Omega$$

with magnetic Neumann boundary condition on $\partial\Omega$

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Semiclassical Magnetic Laplacian

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with magnetic Neumann boundary condition on $\partial\Omega$

- $H(\mathbf{A}, \Omega) = (-i\nabla + \mathbf{A})^2$
- $H_h(\mathbf{A}, \Omega)$: positive and self-adjoint
- If Ω is lipschitz and bounded, $H_h(\mathbf{A}, \Omega)$ has compact resolvent
- If Π is invariant by dilation and $\mathbf{A}$ is linear, by scaling $X = \sqrt{\frac{1}{h}}x$,

$$H_h(\mathbf{A}, \Pi) \equiv h\, H(\mathbf{A}, \Pi)$$

Schrödinger operator

Gauge invariance

Proposition (Gauge invariance)

The eigenvalues of $H_h(\mathbf{A},\Omega)$ only depends on the magnetic field

$$\mathbf{B} = \operatorname{curl} \mathbf{A}$$

Let $\lambda_h(\mathbf{B},\Omega)$ be the smallest eigenvalue of $H_h(\mathbf{A},\Omega)$ and ψ_h a normalized associated eigenvector

$$\begin{cases} (-ih\nabla + \mathbf{A})^2\psi_h = \lambda_h(\mathbf{B},\Omega)\psi_h & \text{in } \Omega \\ \mathbf{n}\cdot(-ih\nabla + \mathbf{A})\psi_h = 0 & \text{on } \partial\Omega \end{cases}$$

Remark. $e^{-i\theta/h}\psi_h$ is an eigenfunction of $H_h(\mathbf{A}+\nabla\theta,\Omega)$ for $\lambda_h(\mathbf{B},\Omega)$

Schrödinger operator

Spectrum

Rayleigh quotients

$$\mathscr{Q}_h[\mathbf{A},\Omega](f) = \frac{q_h[\mathbf{A},\Omega](f)}{\|f\|^2_{\mathrm{L}^2(\Omega)}}, \quad f \in \mathrm{Dom}(q_h[\mathbf{A},\Omega]),\ f \neq 0,$$

where $$q_h[\mathbf{A},\Omega](f) = \|(-ih\nabla + \mathbf{A})f\|^2_{\mathrm{L}^2(\Omega)}$$

Schrödinger operator

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Min-max principle

$$\lambda_h(\mathbf{B},\Omega) = \min\left\{\mathscr{Q}_h[\mathbf{A},\Omega](f),\ f \in \mathrm{Dom}(q_h[\mathbf{A},\Omega]) \setminus \{0\}\right\}$$

Schrödinger operator

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Persson Lemma

$E_{\mathrm{ess}}(\mathbf{B},\Pi)$ bottom of the essential spectrum of $H(\mathbf{A},\Pi)$

$$\Sigma(H(\mathbf{A},\Pi),r) = \inf\left\{\mathscr{Q}[\mathbf{A},\Pi](f), f \in \mathcal{C}_0^\infty(\overline{\Pi} \cap \complement\mathcal{B}_r) \setminus \{0\}\right\}$$

$$\underset{r\to+\infty}{\longrightarrow} E_{\mathrm{ess}}(\mathbf{B},\Pi)$$

Schrödinger operator

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Objectives

determine the influence of the geometry of Ω, the regularity of $\mathbf{A}$ on the asymptotics of $(\lambda_h(\mathbf{B},\Omega),\psi_h)$ as $h \to 0$

2D Schrödinger operator

Polygonal domain

Ω	curvilinear polygonal domain
$\mathfrak{V}$	set of vertices of Ω
$\mathbf{B}$	regular scalar magnetic field
$\mathbf{A}$	associated magnetic potential

Aim: determine the asymptotics of $\lambda_h(\mathbf{B}, \Omega)$

2D Schrödinger operator

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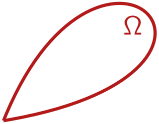
$\underline{\mathbf{B}} = 1$

$\underline{\mathbf{A}} = (0, x_1)$

Aim: determine the asymptotics of $\lambda_h(\mathbf{B}, \Omega)$



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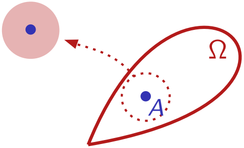

Model	Energy
$H(\underline{\mathbf{A}}, \mathbb{R}^2)$	$E(\underline{\mathbf{B}}, \mathbb{R}^2) = 1$

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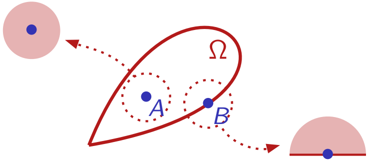

Model	Energy
$H(\underline{\mathbf{A}}, \mathbb{R}^2)$	$E(\underline{\mathbf{B}}, \mathbb{R}^2) = 1$
$H(\underline{\mathbf{A}}, \mathbb{R}^2_+)$	$E(\underline{\mathbf{B}}, \mathbb{R}^2_+) = \Theta_0 \simeq 0.59$

cf. [De Gennes 63, Dauge-Helffer 88, BN12]

2D Schrödinger operator

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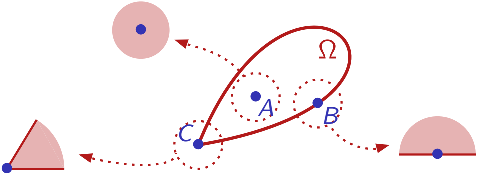

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$H(\underline{\mathbf{A}}, \mathcal{S}_\alpha)$	$E(\underline{\mathbf{B}}, \mathcal{S}_\alpha) = \mu(\alpha)$

cf. [BN03, BN05, BN-Dauge-Martin-Vial 07]

2D Schrödinger operator

Asymptotics

Stratification of $\overline{\Omega}$

$$\overline{\Omega} = \Omega \cup \Big(\bigcup_{\mathbf{e} \in \mathfrak{E}} \mathbf{e} \Big) \cup \Big(\bigcup_{\mathbf{v} \in \mathfrak{V}} \mathbf{v} \Big)$$

with $\mathfrak{E}$ and $\mathfrak{V}$: the set of edges $\mathbf{e}$ and vertices $\mathbf{v}$ of Ω

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	$\mathbf{e}$	half-plane $\mathbb{R}^2_+$	Θ_0
	$\mathbf{v}$	angular sector $\mathcal{S}_\alpha$	$\mu(\alpha(\mathbf{v}))$

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Lowest local energy

$$\mathscr{E}(\mathbf{B}, \Omega) = \inf_{\mathbf{x}\in\overline{\Omega}} E(\mathbf{B}(\mathbf{x}), \Pi_{\mathbf{x}})$$

2D Schrödinger operator

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Lowest local energy

$$\mathscr{E}(\mathbf{B}, \Omega) = \inf_{\mathbf{x}\in\overline{\Omega}} E(\mathbf{B}(\mathbf{x}), \Pi_{\mathbf{x}})$$

Let

$$b = \inf_{\mathbf{x}\in\overline{\Omega}} |\mathbf{B}(\mathbf{x})| \quad \text{and} \quad b' = \inf_{\mathbf{x}\in\partial\Omega} |\mathbf{B}(\mathbf{x})|$$

then

$$\mathscr{E}(\mathbf{B}, \Omega) = \min\Big(b,\ \Theta_0 b',\ \min_{\mathbf{v}\in\mathfrak{V}} |\mathbf{B}(\mathbf{v})| \mu(\alpha(\mathbf{v})) \Big)$$

2D Schrödinger operator

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then

$$\mathscr{E}(\mathbf{B},\Omega) = \min\left(b,\ \Theta_0 b',\ \min_{\mathbf{v}\in\mathfrak{V}} |\mathbf{B}(\mathbf{v})|\mu(\alpha(\mathbf{v}))\right)$$

Theorem

Assume $b \neq 0$, then

$$-Ch^{5/4} \leq \lambda_h(\mathbf{B},\Omega) - h\mathscr{E}(\mathbf{B},\Omega) \leq Ch^{9/8}$$

If $\min_{\mathbf{v}\in\mathfrak{V}} |\mathbf{B}(\mathbf{v})|\mu(\alpha(\mathbf{v})) < min(b, \Theta_0 b')$, *then*

$$|\lambda_h(\mathbf{B},\Omega) - h\mathscr{E}(\mathbf{B},\Omega)| \leq Ch^{3/2}$$

cf. [BN 05]

2D Schrödinger operator

Polygons with straight edges

Theorem

Let $\mathbf{B} = 1$, *then there exist* $C > 0$ *and* $\beta > 0$ *such that*

$$\left| \frac{\lambda_h(B,\Omega)}{h} - \mathscr{E}(B,\Omega) \right| \leq C \exp(-\beta h^{-1/2})$$

cf. [BN-Dauge 06]

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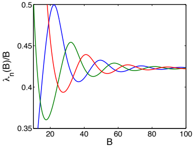
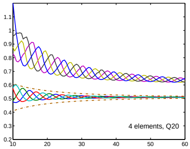
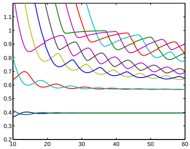


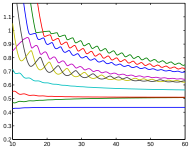
Equilateral triangle



square



rhombus



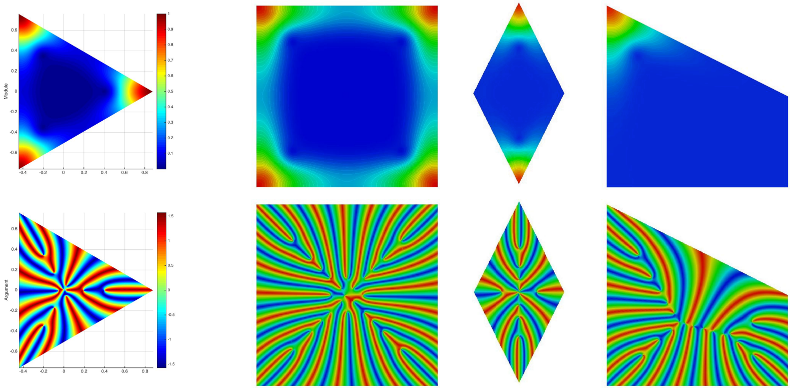
trapezoid

$h^{-1}\lambda_{h,k}$ vs. h^{-1}

cf. [BN-Dauge-Martin-Vial 07]

2D Schrödinger operator

Polygons with straight edges



cf. [BN-Dauge-Martin-Vial 07]

Angular sectors

Eigenvalues

$\mathcal{S}_\alpha = \{\mathbf{x} \in \mathbb{R}^2, x_1 > 0, |x_2| < x_1 \tan \frac{\alpha}{2}\}$ $\qquad$ $\underline{\mathbf{A}}(\mathbf{x}) = (-\frac{1}{2}, \frac{1}{2})$

$H(\underline{\mathbf{A}}, \mathcal{S}_\alpha)$: Neumann realization of $-(i\nabla + \underline{\mathbf{A}})^2$ on $\mathcal{S}_\alpha$

$$\mu_k(\alpha) = \max_{\Psi_1,\dots,\Psi_{k-1}} \min_{\Psi\in[\Psi_1,\dots,\Psi_{k-1}]^\perp} \mathscr{Q}(\underline{\mathbf{A}}, \mathcal{S}_\alpha)$$

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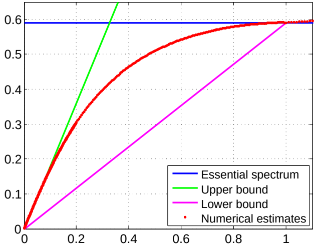
Theorem

1. $E_{\text{ess}}(\underline{\mathbf{B}}, \mathcal{S}_\alpha) = \Theta_0 = \mu_1(\pi)$
   Persson's lemma
   $\Rightarrow$ *half-plane model*

$$E(\underline{\mathbf{B}}, \mathbb{R}^2) = \Theta_0$$

cf. [Jadalalh, Pan, Bo05a, Alouges-BN 06, BN-Dauge-Martin-Vial 07]



$\mu_1(\alpha)$ vs. $\frac{\alpha}{\pi}$, $\frac{\alpha}{\pi} \in (0, 1.2)$

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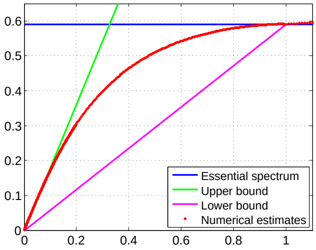
Theorem

1. $E_{\mathrm{ess}}(\underline{\mathbf{B}}, \mathcal{S}_\alpha) = \Theta_0 = \mu_1(\pi)$
2. *For any* $\alpha \in (0, \pi)$,

$$\frac{\Theta_0}{\pi}\alpha < \mu_1(\alpha) \leq \frac{\alpha}{\sqrt{3}}$$

3. *For any* $\alpha \in (0, \frac{\pi}{2}]$, $\mu_1(\alpha) < \Theta_0$



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Angular sectors

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$\mathcal{S}_\alpha = \{\mathbf{x} \in \mathbb{R}^2, x_1 > 0, |x_2| < x_1 \tan \frac{\alpha}{2}\}$ $\qquad$ $\underline{\mathbf{A}}(\mathbf{x}) = (-\frac{1}{2}, \frac{1}{2})$

$H(\underline{\mathbf{A}}, \mathcal{S}_\alpha)$: Neumann realization of $-(i\nabla + \underline{\mathbf{A}})^2$ on $\mathcal{S}_\alpha$

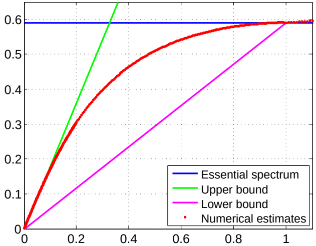
$$\mu_k(\alpha) = \max_{\Psi_1,\dots,\Psi_{k-1}} \min_{\Psi \in [\Psi_1,\dots,\Psi_{k-1}]^\perp} \mathscr{Q}(\underline{\mathbf{A}}, \mathcal{S}_\alpha)$$

Conjecture

$\alpha \mapsto \mu_1(\alpha)$ is

increasing from $(0, \pi]$ onto $(0, \Theta_0]$

equal to Θ_0 for $\alpha \in [\pi, 2\pi)$



$\mu_1(\alpha)$ vs. $\frac{\alpha}{\pi}$, $\frac{\alpha}{\pi} \in (0, 1.2)$

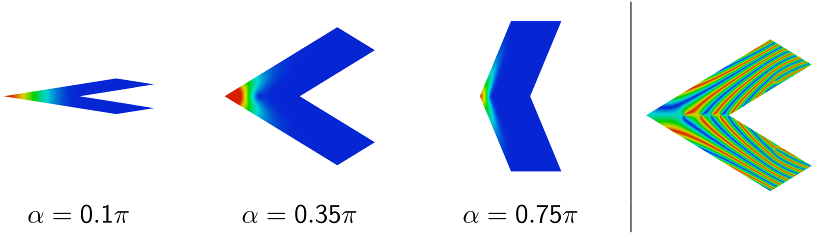
Angular sectors

Eigenvectors

Theorem

Let $\alpha \in (0, 2\pi)$ and $k \geq 1$ be such that $\mu_k(\alpha) < \Theta_0$
Let Ψ_k^α be a normalized eigenfunction associated with $\mu_k(\alpha)$. Then

$$\forall \varepsilon > 0, \exists C_{\varepsilon,\alpha}, \quad \left\| \exp\left(\left(\sqrt{\Theta_0 - \mu_k(\alpha)} - \varepsilon\right) |\mathbf{X}|\right) \Psi_k^\alpha \right\|_{\mathrm{H}^1_{\underline{\mathbf{A}}}(\mathcal{S}_\alpha)} \leq C_{\varepsilon,\alpha}$$



$\alpha = 0.1\pi$ $\alpha = 0.35\pi$ $\alpha = 0.75\pi$

cf. [Bo05a,BN-Dauge-Martin-Vial 07]

Angular sectors

Small angles

Theorem

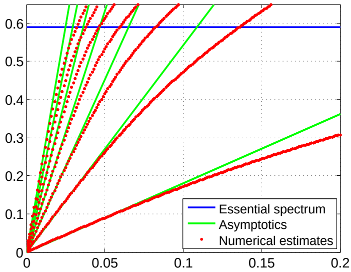
1. *For any n, there exists $(m_j)_{j\leq n}$ such that*

$$\mu_1(\alpha) = \frac{\alpha}{\sqrt{3}} + \sum_{j=1}^{n} m_j \alpha^{2j+1} + \mathcal{O}_n(\alpha^{2n+3}) \qquad \textit{as} \qquad \alpha \to 0$$

2. *For any $k \geq 1$,*

$$\lim_{\alpha\to 0} \frac{\mu_k(\alpha)}{\alpha} = \frac{2k-1}{\sqrt{3}}$$

cf. [Bo05a,BDMV07]



$\mu_k(\alpha)$ *vs.* $\frac{\alpha}{\pi}$, $k = 1, \dots, 6$, $\frac{\alpha}{\pi} \in (0, 0.2)$

Taxonomy

Tangent cones

Stratification of $\overline{\Omega}$

$$\overline{\Omega} = \Omega \cup \Big(\bigcup_{\mathbf{f} \in \mathfrak{F}} \mathbf{f} \Big) \cup \Big(\bigcup_{\mathbf{e} \in \mathfrak{E}} \mathbf{e} \Big) \cup \Big(\bigcup_{\mathbf{v} \in \mathfrak{V}} \mathbf{v} \Big)$$

with $\mathfrak{F}$, $\mathfrak{E}$ and $\mathfrak{V}$: the set of faces $\mathbf{f}$, edges $\mathbf{e}$ and vertices $\mathbf{v}$ of Ω

For any $\mathbf{x} \in \overline{\Omega}$, let $\Pi_{\mathbf{x}}$ be its tangent cone

Dimension	$\mathbf{x} \in \overline{\Omega}$	Model geometry for $\Pi_{\mathbf{x}}$
2D	Ω	plane $\mathbb{R}^2$
	$\mathbf{e}$	half-plane $\mathbb{R}^2_+$
	$\mathbf{v}$	angular sector $\mathcal{S}_\alpha$
3D	Ω	space $\mathbb{R}^3$
	$\mathbf{f}$	half-space $\mathbb{R}^3_+$
	$\mathbf{e}$	infinite wedge $\mathcal{W}_\alpha = \mathcal{S}_\alpha \times \mathbb{R}$
	$\mathbf{v}$	3d cone $\mathfrak{C}$

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	$\mathbf{v}$	3d cone $\mathfrak{C}$

Remark

- *Ω polyhedral: all the tangent cones are straight (no curvature)*
- *More general corner domains: the tangent cone has curvature (unbounded)*
  Example : circular cone

Taxonomy

Local ground energy

$\mathbf{x}$ $\in \overline{\Omega}$

$\Pi_{\mathbf{x}}$ tangent cone at $\mathbf{x}$

$\mathbf{B}_{\mathbf{x}}$ magnetic field $\mathbf{B}$ frozen at $\mathbf{x}$

$\mathbf{A}_{\mathbf{x}}$ linear approximation of $\mathbf{A}$ at $\mathbf{x}$ so that $\operatorname{curl} \mathbf{A}_{\mathbf{x}} = \mathbf{B}_{\mathbf{x}}$

Remark. By scaling $Y = \sqrt{\frac{1}{h}}\mathbf{y}$,

$$H_h(\mathbf{A}_{\mathbf{x}}, \Pi_{\mathbf{x}}) \;\equiv\; h\; H(\mathbf{A}_{\mathbf{x}}, \Pi_{\mathbf{x}})$$

Taxonomy

Local ground energy

$\mathbf{x}$	$\in \overline{\Omega}$
$\Pi_{\mathbf{x}}$	tangent cone at $\mathbf{x}$
$\mathbf{B}_{\mathbf{x}}$	magnetic field $\mathbf{B}$ frozen at $\mathbf{x}$
$\mathbf{A}_{\mathbf{x}}$	linear approximation of $\mathbf{A}$ at $\mathbf{x}$ so that $\operatorname{curl} \mathbf{A}_{\mathbf{x}} = \mathbf{B}_{\mathbf{x}}$

Remark. By scaling $Y = \sqrt{\frac{1}{h}}\mathbf{y}$,

$$H_h(\mathbf{A}_{\mathbf{x}}, \Pi_{\mathbf{x}}) \equiv h\, H(\mathbf{A}_{\mathbf{x}}, \Pi_{\mathbf{x}})$$

Definition (Local ground energy)

$E(\mathbf{B}_{\mathbf{x}}, \Pi_{\mathbf{x}})$: bottom of the spectrum of the tangent operator $H(\mathbf{A}_{\mathbf{x}}, \Pi_{\mathbf{x}})$

Taxonomy

Local ground energy

$\mathbf{x}$ $\in \overline{\Omega}$

$\Pi_{\mathbf{x}}$ tangent cone at $\mathbf{x}$

$\mathbf{B}_{\mathbf{x}}$ magnetic field $\mathbf{B}$ frozen at $\mathbf{x}$

$\mathbf{A}_{\mathbf{x}}$ linear approximation of $\mathbf{A}$ at $\mathbf{x}$ so that $\operatorname{curl} \mathbf{A}_{\mathbf{x}} = \mathbf{B}_{\mathbf{x}}$

Definition (Local ground energy)

$E(\mathbf{B}_{\mathbf{x}}, \Pi_{\mathbf{x}})$: bottom of the spectrum of the tangent operator $H(\mathbf{A}_{\mathbf{x}}, \Pi_{\mathbf{x}})$

We define the lowest local energy

$$\mathscr{E}(\mathbf{B}, \Omega) := \inf_{\mathbf{x} \in \overline{\Omega}} E(\mathbf{B}_{\mathbf{x}}, \Pi_{\mathbf{x}})$$

3D Schrödinger operator

Polyhedral and corner domains

Theorem

Let Ω, $\mathbf{A}$ *such that* $\mathbf{B} = \operatorname{curl} \mathbf{A}$ *does not vanish on* $\overline{\Omega}$
Then there exist $C(\Omega) > 0$ *and* $h_0 > 0$ *such that* $\forall h \in (0, h_0)$

- *If* $\mathbf{A} \in W^{2,\infty}(\Omega)$

$$\left|\lambda_h(\mathbf{B},\Omega) - h\mathscr{E}(\mathbf{B},\Omega)\right| \leq \begin{cases} C_\Omega\left(1 + \|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)}\right) h^{\frac{11}{10}}, & \Omega \text{ corner domain} \\ C_\Omega\left(1 + \|\mathbf{A}\|^2_{W^{2,\infty}(\Omega)}\right) h^{\frac{5}{4}}, & \Omega \text{ polyhedral domain} \end{cases}$$

cf. [BN-Dauge-Popoff 16]

3D Schrödinger operator

Polyhedral and corner domains

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- *If* $\mathbf{A} \in W^{3,\infty}(\Omega)$

$$\lambda_h(\mathbf{B},\Omega) - h\mathscr{E}(\mathbf{B},\Omega) \leq \begin{cases} C_\Omega\left(1 + \|\mathbf{A}\|^2_{W^{3,\infty}(\Omega)}\right) h^{\frac{9}{8}}, & \Omega \text{ corner domain} \\ C_\Omega\left(1 + \|\mathbf{A}\|^2_{W^{3,\infty}(\Omega)}\right) h^{\frac{4}{3}}, & \Omega \text{ polyhedral domain} \end{cases}$$

cf. [BN-Dauge-Popoff 16]

3D Schrödinger operator

Polyhedral and corner domains

Theorem

$E_{\mathrm{ess}}(\mathbf{B},\Pi_{\mathbf{x}})$ *bottom of the essential spectrum of* $H(\mathbf{A},\Pi_{\mathbf{x}})$

- *If there exists a vertex* $\mathbf{v}$ *of* Ω *such that*

$$\mathscr{E}(\mathbf{B},\Omega) = E(\mathbf{B},\Pi_{\mathbf{v}}) < E_{\mathrm{ess}}(\mathbf{B},\Pi_{\mathbf{v}}),$$

then $\lambda_h(\mathbf{B},\Omega) \leq h\mathscr{E}(\mathbf{B},\Omega) + C\,h^{3/2}|\log h|$

cf. [BN-Dauge-Popoff 16]

3D Schrödinger operator

Polyhedral and corner domains

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then $\lambda_h(\mathbf{B},\Omega) \le h\mathscr{E}(\mathbf{B},\Omega) + C\,h^{3/2}|\log h|$

- *If the lowest local energy is attained at vertices only, i.e.*

$$\mathscr{E}(\mathbf{B},\Omega) < \inf_{\mathbf{x}\in\overline{\Omega}\setminus\mathfrak{V}} E(\mathbf{B},\Pi_{\mathbf{x}}),$$

then $|\lambda_h(\mathbf{B},\Omega) - h\mathscr{E}(\mathbf{B},\Omega)| \le C\,h^{3/2}$ *as* $h\to 0$

and the corresponding eigenfunction concentrates near the vertices $\mathbf{v}$ *such that* $\mathscr{E}(\mathbf{B},\Omega) = E(\mathbf{B},\Pi_{\mathbf{v}})$

cf. [BN-Dauge-Popoff 16]

Space and half-space

- **Full space**

$$\Pi = \mathbb{R}^3 \quad \text{and} \quad \underline{\mathbf{B}} = (1, 0, 0)$$

Then

$$E(\underline{\mathbf{B}}, \mathbb{R}^3) = 1$$

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Then

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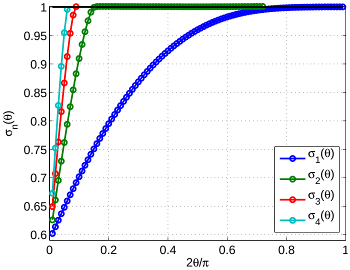
- **Half-space**

$$\Pi = \mathbb{R}^2 \times \mathbb{R}_+ = \{\mathbf{x} \in \mathbb{R}^3, \ x_3 > 0\}$$
$$\underline{\mathbf{B}}_\theta = (0, \cos\theta, \sin\theta)$$

Then

$$E(\underline{\mathbf{B}}_\theta, \mathbb{R}^2 \times \mathbb{R}_+) = \sigma(\theta)$$

$\theta \mapsto \sigma(\theta)$ continuous and increasing on $[0, \frac{\pi}{2}]$ with $\sigma(0) = \Theta_0$ and $\sigma(\frac{\pi}{2}) = 1$



cf. [Lu-Pan 00, Helffer-Morame 02, BDPR12]

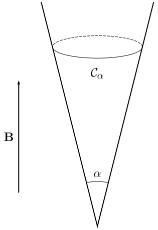
Circular cones

Magnetic field along the axis

$\mathcal{C}_\alpha^\circ = \{\mathbf{x} \in \mathbb{R}^3,\ x_3 > 0,\ x_1^2 + x_2^2 < x_3^2 \tan^2 \alpha\}$

$\underline{\mathbf{B}}(\mathbf{x}) = (0,0,1), \qquad \underline{\mathbf{A}}(\mathbf{x}) = \frac{1}{2}(-x_2, x_1, 0)$

$E_n(\underline{\mathbf{B}}, \mathcal{C}_\alpha^\circ)$ n-th eigenvalue of $H(\underline{\mathbf{A}}, \mathcal{C}_\alpha^\circ)$



Circular cones

Magnetic field along the axis

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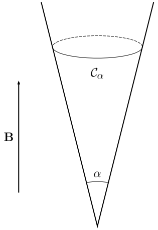


Theorem

- *Essential spectrum*

$$E_{\mathrm{ess}}(\underline{\mathbf{A}}, \mathcal{C}_\alpha^\circ) = \sigma(\alpha) \quad \in (\Theta_0, 1]$$

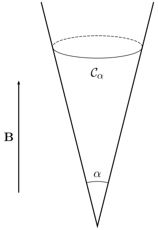
Persson's lemma ⇒ half-space model

$$E(\underline{\mathbf{B}}_\theta, \mathbb{R}^2 \times \mathbb{R}_+) = \sigma(\alpha)$$

cf. [BN-Raymond 13]

Circular cones

Magnetic field along the axis



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Theorem

- *Essential spectrum*

$$E_{\mathrm{ess}}(\underline{\mathbf{A}}, \mathcal{C}_\alpha^\circ) = \sigma(\alpha) \quad \in (\Theta_0, 1]$$

- *Eigenvalues*

$$E_n(\underline{\mathbf{B}}, \mathcal{C}_\alpha^\circ) = \frac{4n-1}{4\sqrt{2}}\alpha + \mathcal{O}(\alpha^3) \qquad \textit{as} \quad \alpha \to 0$$

Circular cones

Change of variables

Using a dilatation in spherical coordinates

$$(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \alpha^{-1/2}(t \cos\theta \sin\alpha\varphi,\ \ t \sin\theta \sin\alpha\varphi,\ \ t \cos\alpha\varphi)$$

Circular cones

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we consider the operator $\mathfrak{M}^\alpha$

$$\mathfrak{M}^\alpha = \frac{1}{t^2}D_t t^2 D_t + \frac{1}{t^2\sin^2\alpha\varphi}\left(D_\theta + \frac{\sin^2\alpha\varphi}{2\alpha}t^2\right)^2 + \frac{1}{\alpha^2\ t^2\sin\alpha\varphi}D_\varphi\sin\alpha\varphi D_\varphi$$

on $\mathrm{L}^2(\mathcal{P}, \mathrm{d}\hat{\mu})$ with $\mathrm{d}\hat{\mu} = t^2\sin\alpha\varphi\,\mathrm{d}t\,\mathrm{d}\theta\,\mathrm{d}\varphi$ and

$$\mathcal{P} = \{(t,\theta,\varphi)\in\mathbb{R}^3,\ \ t>0,\ \ \theta\in(0,2\pi),\ \ \varphi\in(0,\tfrac{1}{2})\}$$

Circular cones

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$$\mathcal{P} = \{(t,\theta,\varphi)\in\mathbb{R}^3,\ \ t>0,\ \ \theta\in(0,2\pi),\ \ \varphi\in(0,\tfrac{1}{2})\}$$

We have

$$\mathrm{sp}(H(\underline{\mathbf{A}},\mathcal{C}_\alpha^\circ)) = \alpha\,\mathrm{sp}(\mathfrak{M}^\alpha)$$

Circular cones

Change of variables

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$$(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \alpha^{-1/2}(t \cos\theta \sin\alpha\varphi,\ \ t \sin\theta \sin\alpha\varphi,\ \ t\cos\alpha\varphi)$$

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Using the Fourier series, we consider the family of 2D-operators, $m \in \mathbb{Z}$

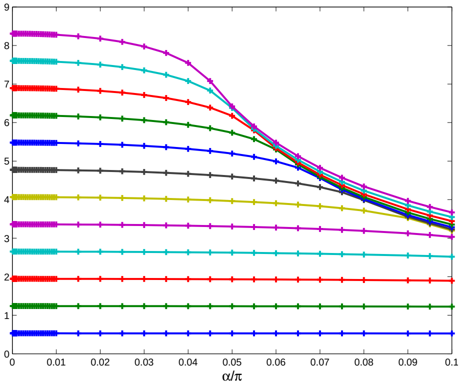
$$\mathfrak{M}_{\alpha,m} = \frac{1}{t^2} D_t t^2 D_t + \frac{1}{t^2 \sin^2 \alpha\varphi} \left(m + \frac{\sin^2 \alpha\varphi}{2\alpha} t^2 \right)^2 + \frac{1}{\alpha^2\ \ t^2 \sin\alpha\varphi} D_\varphi \sin\alpha\varphi D_\varphi$$

on $\mathrm{L}^2(\mathcal{R},\, \mathrm{d}\mu)$, $\ \mathrm{d}\mu = t^2 \sin\alpha\varphi \,\mathrm{d}t\, \mathrm{d}\varphi$ and

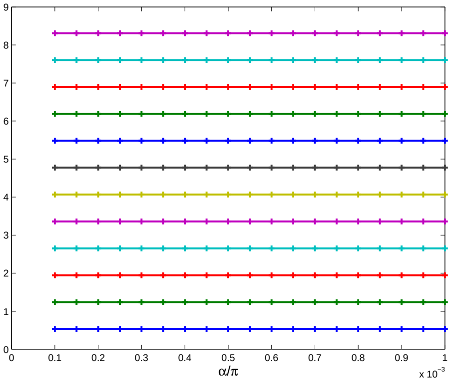
$$\mathcal{R} = \{(t,\varphi) \in \mathbb{R}^2,\ \ t > 0,\ \ \varphi \in (0, \tfrac{1}{2})\}$$

Circular cones

Numerical simulations



$\lambda_n(\alpha, 0)$ vs. $\frac{\alpha}{\pi}$, $\frac{\alpha}{\pi} \in (0, 0.1)$



$\lambda_n(\alpha, 0)$ vs. $\frac{\alpha}{\pi}$, $\frac{\alpha}{\pi} \in \{\frac{k}{10^4}, 1 \leq k \leq 10\}$

Circular cones

Asymptotic expansion

$$\mathfrak{M}_{\alpha,m} = \frac{1}{t^2} D_t t^2 D_t + \frac{1}{t^2 \sin^2 \alpha\varphi} \left(m + \frac{\sin^2 \alpha\varphi}{2\alpha} t^2 \right)^2 + \frac{1}{\alpha^2 \ t^2 \sin \alpha\varphi} D_\varphi \sin \alpha\varphi D_\varphi$$

We first look for quasi-modes for $\mathfrak{M}_{\alpha,0}$.

Circular cones

Asymptotic expansion

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$$\mathfrak{M}_{\alpha,0} \sim \alpha^{-2}\mathfrak{M}_{-1} + \alpha^{0}\mathfrak{M}_0 + \sum_{j\geq 1} \alpha^{2j}\mathfrak{M}_j$$

where

$$\mathfrak{M}_{-1} = -\frac{1}{t^2\varphi}\partial_\varphi \varphi \partial_\varphi, \quad \mathfrak{M}_0 = -\frac{1}{t^2}\partial_t t^2 \partial_t + \frac{\varphi^2 t^2}{4} + \frac{1}{3t^2}\varphi\partial_\varphi$$

Circular cones

Asymptotic expansion

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We look for quasi-eigenpairs expressed as formal series

$$\psi \sim \sum_{j\geq 0} \alpha^{2j}\psi_j, \qquad \lambda \sim \alpha^{-2}\lambda_{-1} + \alpha^{0}\lambda_0 + \sum_{j\geq 1}\alpha^{2j}\lambda_j$$

so that, formally, we have

$$\mathfrak{M}_{\alpha,0}\psi \sim \lambda\psi$$

Circular cones

Asymptotic expansion

Theorem

For all $n \geq 1$, there exist $\alpha_0(n) > 0$ and a sequence $(\gamma_{j,n})_{j\geq 0}$ such that, for all $\alpha \in (0, \alpha_0(n))$, the n-th eigenvalue exists and satisfies

$$E_n(\underline{\mathbf{B}}, \mathcal{C}_\alpha^\circ) \underset{\alpha\to 0}{\sim} \alpha \sum_{j\geq 0} \gamma_{j,n} \alpha^{2j}, \qquad \gamma_{0,n} = \frac{4n-1}{2^{5/2}}$$

+ *asymptotics expansion for the eigenfunctions*

cf. [BN-Raymond 13 & 15]

Circular cones

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Remark

$$0.530 \simeq \frac{3}{2^{5/2}} = \lim_{\alpha\to 0} \frac{E(\underline{\mathbf{B}}, \mathcal{C}_\alpha^\circ)}{\alpha} < \lim_{\alpha\to 0} \frac{E(B, \mathcal{S}_\alpha)}{\alpha} = \frac{1}{\sqrt{3}} \simeq 0.577$$

cf. [BN-Raymond 13 & 15]

Circular cones

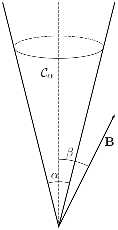
Influence of $\underline{\mathbf{B}}$

$$\mathcal{C}_\alpha^\circ = \{\mathbf{x} \in \mathbb{R}^3,\ x_3 > 0,\ x_1^2 + x_2^2 < x_3^2 \tan^2 \alpha\}$$

$$\underline{\mathbf{B}}_\beta = (0, \sin\beta, \cos\beta)$$

$$\underline{\mathbf{A}}_\beta = \tfrac{1}{2}(x_3 \sin\beta - x_2 \cos\beta, x_1 \cos\beta, -x_1 \sin\beta)$$

$E_n(\underline{\mathbf{B}}_\beta, \mathcal{C}_\alpha^\circ)$ n-th eigenvalue of $H(\underline{\mathbf{A}}_\beta, \mathcal{C}_\alpha^\circ)$



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Circular cones

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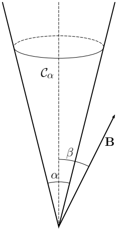

Theorem

- $E_{\mathrm{ess}}(\underline{\mathbf{A}}_\beta, \mathcal{C}_\alpha^\circ) \in (\Theta_0, 1]$
- *For all $n \geq 1$, there exist $\alpha_0(n) > 0$ and a sequence $(\gamma_{j,n}^\beta)_{j\geq 0}$ such that, for all $\alpha \in (0, \alpha_0(n))$, the n-th eigenvalue exists and satisfies*

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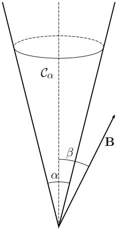
Circular cones

Influence of $\underline{\mathbf{B}}$



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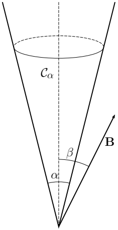

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$\gamma_{0,n}^\beta$ minimal when $\beta = 0$ and in that case, $\gamma_{2j-1,n}^0 = 0$

cf. [BN-Raymond 15]

3D-cones

Notations

ω bounded and connected open subset of $\mathbb{R}^2$

$\omega_\varepsilon = \varepsilon\omega$ sharp cone ($\varepsilon > 0$)

$\mathcal{C}_\omega$ cone defined by

$$\mathcal{C}_\omega = \left\{ \mathbf{x} = (\mathsf{x}_1, \mathsf{x}_2, \mathsf{x}_3) \in \mathbb{R}^3 : \quad \mathsf{x}_3 > 0 \quad \text{and} \quad \left(\frac{\mathsf{x}_1}{\mathsf{x}_3}, \frac{\mathsf{x}_2}{\mathsf{x}_3}\right) \in \omega \right\}$$

$\mathbf{B} = (\mathrm{B}_1, \mathrm{B}_2, \mathrm{B}_3)^T$ constant magnetic field

$\mathbf{A}$ associated linear magnetic potential, i.e., such that $\operatorname{curl} \mathbf{A} = \mathbf{B}$

n-th Rayleigh quotient of $H(\mathbf{A}, \mathcal{C}_\omega)$

$$E_n(\mathbf{B}, \mathcal{C}_\omega) = \sup_{u_1,\ldots,u_{n-1} \in \mathrm{Dom}(q[\mathbf{A},\mathcal{C}_\omega])} \ \inf_{\substack{u \in [u_1,\ldots,u_{n-1}]^\perp \\ u \in \mathrm{Dom}(q[\mathbf{A},\mathcal{C}_\omega])}} \mathscr{Q}[\mathbf{A}, \mathcal{C}_\omega](u)$$

Schrödinger operator on 3D-cones

Upper-bound of the Rayleigh quotients

Theorem

Normalized moments:

$$m_k = \frac{1}{|\omega|}\int_\omega \mathrm{x}_1^k \mathrm{x}_2^{2-k}\,\mathrm{d}\mathrm{x}_1\mathrm{d}\mathrm{x}_2$$

Then

$$E_n(\mathbf{B},\omega) \leq (4n-1)e(\mathbf{B},\omega)$$

with

$$e(\mathbf{B},\omega) = \left(\mathrm{B}_3^2\frac{m_0 m_2 - m_1^2}{m_0+m_2} + \mathrm{B}_2^2 m_2 + \mathrm{B}_1^2 m_0 - 2\mathrm{B}_1\mathrm{B}_2 m_1\right)^{1/2}$$

cf. [BN-Dauge-Popoff-Raymond 15]

Schrödinger operator on 3D-cones

Upper-bound of the Rayleigh quotients

Examples.

1. Circular cone $\omega = \mathcal{B}(0, \rho)$ Thus $m_0 = m_2 = \frac{|\omega|}{4\pi}$ and $m_1 = 0$

$$E_n(\mathbf{B}, \mathcal{C}_\omega) \leq (4n-1)e(\mathbf{B}, \omega) = \frac{4n-1}{2}\sqrt{\frac{|\omega|}{\pi}}\left(\frac{\mathrm{B}_3^2}{2} + \mathrm{B}_1^2 + \mathrm{B}_2^2\right)^{1/2}$$

Schrödinger operator on 3D-cones

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Examples.

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If $\mathbf{B} = (0,\sin\beta,\cos\beta)^\mathsf{T}$ and $\rho = \tan\frac{\alpha}{2}$, thus $\mathcal{C}_\omega = \mathcal{C}^\circ_\alpha$ and

$$\forall\alpha\in(0,\pi),\quad E_n(\mathbf{B},\mathcal{C}^\circ_\alpha) \leq \frac{4n-1}{2^{3/2}}\tan\frac{\alpha}{2}\sqrt{1+\sin^2\beta}$$

in coherence with the asymptotics as $\alpha\to 0$ in [BN-Raymond 13 & 15]

Schrödinger operator on 3D-cones

Upper-bound of the Rayleigh quotients

Examples.

1. Circular cone $\omega = \mathcal{B}(0,\rho)$

$$E_n(\mathbf{B},\mathcal{C}_\omega) \leq (4n-1)e(\mathbf{B},\omega) = \frac{4n-1}{2}\sqrt{\frac{|\omega|}{\pi}}\left(\frac{\mathrm{B}_3^2}{2}+\mathrm{B}_1^2+\mathrm{B}_2^2\right)^{1/2}$$

2. Rectangle $\omega = [-\ell,\ell]\times[-L,L]$

$$E_n(\mathbf{B},\mathcal{C}_\omega) \leq \frac{4n-1}{\sqrt{3}}\left(\mathrm{B}_3^2\frac{\ell^2L^2}{\ell^2+L^2}+\mathrm{B}_1^2L^2+\mathrm{B}_2^2\ell^2\right)^{1/2}$$

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Upper-bound of the Rayleigh quotients

Examples.

1. Circular cone $\omega = \mathcal{B}(0,\rho)$

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2. Rectangle $\omega = [-\ell,\ell]\times[-L,L]$
3. Square $\omega = [-\ell,\ell]^2$

$$E_n(\mathbf{B},\mathcal{C}_\omega) \leq \frac{4n-1}{2}\frac{\sqrt{|\omega|}}{\sqrt{3}}\left(\frac{\mathrm{B}_3^2}{2}+\mathrm{B}_1^2+\mathrm{B}_2^2\right)^{1/2}$$

Schrödinger operator on 3D-cones

Upper-bound of the Rayleigh quotients

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disc	square
$\frac{1}{\sqrt{\pi}} \simeq 0.5642$	$\frac{1}{\sqrt{3}} \simeq 0.5774$

Schrödinger operator on 3D-cones

Upper-bound of the Rayleigh quotients

Let

$$\mathcal{A}(\mathbf{B}) = \{\mathbf{A} \in \mathcal{L}(\mathbb{R}^3) : \quad \partial_{x_3}\mathbf{A} = 0 \quad \text{and} \quad \nabla \times \mathbf{A} = \mathbf{B}\}$$

Let φ real valued and only dependent on the x_3 variable, then

$$q[\mathbf{A}, \mathcal{C}_\omega](\varphi) = \int_{\mathcal{C}_\omega} |\mathbf{A}(\mathbf{x})|^2 |\varphi(x_3)|^2 + |\partial_{x_3}\varphi(x_3)|^2 \mathrm{d}\mathbf{x}$$

Schrödinger operator on 3D-cones

Upper-bound of the Rayleigh quotients

Let

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Change of variables $\mathbf{X} = (X_1, X_2, X_3) = \left(\frac{x_1}{x_3}, \frac{x_2}{x_3}, x_3\right)$

$$q[\mathbf{A}, \mathcal{C}_\omega](\varphi) = \int_{\omega \times \mathbb{R}^+} \left(|\mathbf{A}(\mathbf{X})|^2 X_3^2 |\varphi(X_3)|^2 + |\varphi'(X_3)|^2\right) X_3^2 \mathrm{d}\mathbf{X}$$

$$= |\omega| \int_{\mathbb{R}^+} |\varphi'(X_3)|^2 X_3^2 \mathrm{d}X_3 + \|\mathbf{A}\|^2_{\mathrm{L}^2(\omega)} \int_{\mathbb{R}^+} |\varphi(X_3)|^2 X_3^4 \mathrm{d}X_3$$

$$\|\varphi\|^2_{\mathrm{L}^2(\mathcal{C}_\omega)} = |\omega| \int_{\mathbb{R}^+} |\varphi(X_3)|^2 X_3^2 \mathrm{d}X_3$$

Schrödinger operator on 3D-cones

Upper-bound of the Rayleigh quotients

Weighted space $\mathrm{L}^2_{\mathrm{w}}(\mathbb{R}_+) := \mathrm{L}^2(\mathbb{R}_+, \mathrm{x}^2 \mathrm{d}\mathrm{x})$

Quadratic form $\mathfrak{p}[\lambda](u) = \int_{\mathbb{R}_+} \left(|u'(\mathrm{x})|^2 + \lambda \mathrm{x}^2 |u(\mathrm{x})|^2\right) \mathrm{x}^2 \mathrm{d}\mathrm{x}$

on $\mathrm{B}_{\mathrm{w}}(\mathbb{R}_+) := \{u \in \mathrm{L}^2_{\mathrm{w}}(\mathbb{R}_+) : \mathrm{x}u \in \mathrm{L}^2_{\mathrm{w}}(\mathbb{R}_+), u' \in \mathrm{L}^2_{\mathrm{w}}(\mathbb{R}_+)\}$

Let $\mathbf{A} \in \mathcal{A}(\mathbf{B})$ and $\varphi \in \mathrm{B}_{\mathrm{w}}(\mathbb{R}_+)$

Schrödinger operator on 3D-cones

Upper-bound of the Rayleigh quotients

Weighted space $\mathrm{L}^2_{\mathrm{w}}(\mathbb{R}_+) := \mathrm{L}^2(\mathbb{R}_+, \mathrm{x}^2\mathrm{dx})$

Quadratic form $\mathfrak{p}[\lambda](u) = \int_{\mathbb{R}_+} \left(|u'(\mathrm{x})|^2 + \lambda \mathrm{x}^2 |u(\mathrm{x})|^2\right) \mathrm{x}^2\mathrm{dx}$

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Let $\mathbf{A} \in \mathcal{A}(\mathbf{B})$ and $\varphi \in \mathrm{B}_{\mathrm{w}}(\mathbb{R}_+)$

Then

$$\frac{q[\mathbf{A}, \mathcal{C}_\omega](\varphi)}{\|\varphi\|^2_{\mathrm{L}^2(\mathcal{C}_\omega)}} = \frac{\mathfrak{p}[\lambda](\varphi)}{\|\varphi\|^2_{\mathrm{L}^2_{\mathrm{w}}(\mathbb{R}_+)}} \qquad \text{with} \qquad \lambda = \frac{\|\mathbf{A}\|^2_{\mathrm{L}^2(\omega)}}{|\omega|}$$

Schrödinger operator on 3D-cones

Upper-bound of the Rayleigh quotients

$$\frac{q[\mathbf{A},\mathcal{C}_\omega](\varphi)}{\|\varphi\|^2_{\mathrm{L}^2(\mathcal{C}_\omega)}} = \frac{\mathfrak{p}\left[\lambda\right](\varphi)}{\|\varphi\|^2_{\mathrm{L}^2_{\mathrm{w}}(\mathbb{R}_+)}} \quad \text{with} \quad \lambda = \frac{\|\mathbf{A}\|^2_{\mathrm{L}^2(\omega)}}{|\omega|}$$

Change of function $u \mapsto U := \mathrm{x}u$ to eliminate the weight

$$\mathfrak{p}[\lambda](u) = \int_{\mathbb{R}_+} \left(|U'(\mathrm{x})|^2 + \lambda \mathrm{x}^2 |U(\mathrm{x})|^2\right) \mathrm{dx} \quad \text{and} \quad \|u\|^2_{\mathrm{L}^2_{\mathrm{w}}(\mathbb{R}_+)} = \|U\|^2_{\mathrm{L}^2(\mathbb{R}_+)}$$

$\Rightarrow$ harmonic oscillator on $\mathbb{R}_+$ with Dirichlet condition at 0

Proposition

Let **B** *be a constant magnetic field. Then for all $n \in \mathbb{N}^*$, we have*

$$E_n(\mathbf{B},\mathcal{C}_\omega) \leq \frac{4n-1}{\sqrt{|\omega|}} \inf_{\mathbf{A}\in\mathcal{A}(\mathbf{B})} \|\mathbf{A}\|_{\mathrm{L}^2(\omega)}$$

Schrödinger operator on 3D-cones

Upper-bound of the Rayleigh quotients

By linearity of $\mathbf{A} \in \mathcal{A}(\mathbf{B})$, we have necessarily $A_3(\mathbf{x}) = B_1x_2 - B_2x_1$
Therefore

$$\inf_{\mathbf{A}\in\mathcal{A}(\mathbf{B})} \|\mathbf{A}\|_{L^2(\omega)} = \left(B_3^2 \inf_{\mathbf{A}'\in\mathcal{A}'} \|\mathbf{A}'\|^2_{L^2(\omega)} + \int_\omega (B_1x_2 - B_2x_1)^2 dx_1 dx_2 \right)^{1/2}$$

with

$$\mathcal{A}' = \{\mathbf{A}' \in \mathcal{L}(\mathbb{R}^2) : \quad \nabla_{x_1,x_2} \times \mathbf{A}' = 1\}$$

Schrödinger operator on 3D-cones

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Proposition

$$\inf_{\mathbf{A}'\in\mathcal{A}'} \|\mathbf{A}'\|^2_{L^2(\omega)} = \frac{M_0 M_2 - M_1^2}{M_0 + M_2} \quad \textit{with } M_k := \int_\omega x_1^k x_2^{2-k} \mathrm{d}x_1 \mathrm{d}x_2$$

The minimizer is unique and given by

$$\mathbf{A}'_0(x_1, x_2) = \frac{1}{M_0 + M_2} \begin{pmatrix} M_1 & -M_0 \\ M_2 & -M_1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Schrödinger operator on 3D-cones

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Proposition

$$E_n(\mathbf{B},\omega) \leq (4n-1)e(\mathbf{B},\omega)$$

with

$$e(\mathbf{B},\omega) = \left(B_3^2 \frac{m_0 m_2 - m_1^2}{m_0 + m_2} + B_2^2 m_2 + B_1^2 m_0 - 2B_1 B_2 m_1 \right)^{\frac{1}{2}}$$

$$m_k = \frac{1}{|\omega|} \int_\omega x_1^k x_2^{2-k} \mathrm{d}x_1 \mathrm{d}x_2$$

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with

$$\mathcal{A}' = \{\mathbf{A}' \in \mathcal{L}(\mathbb{R}^2) : \quad \nabla_{x_1,x_2} \times \mathbf{A}' = 1\}$$

Proposition

$$E_n(\mathbf{B}, \omega_\varepsilon) \leq (4n-1) e(\mathbf{B}, \omega)\, \varepsilon$$

with

$$e(\mathbf{B},\omega) = \left(B_3^2 \frac{m_0 m_2 - m_1^2}{m_0 + m_2} + B_2^2 m_2 + B_1^2 m_0 - 2B_1 B_2 m_1 \right)^{\frac{1}{2}}$$

$$m_k = \frac{1}{|\omega|} \int_\omega x_1^k x_2^{2-k} \mathrm{d}x_1 \mathrm{d}x_2$$

Schrödinger operator on 3D-cones

Essential spectrum

$\widehat{\omega}$ cylinder $\omega \times \mathbb{R}$

$\widehat{\Pi}_{\mathbf{x}}$ tangent cone to $\widehat{\omega}$ at $\mathbf{x}$

Ground energy on substructures

$$\mathscr{E}(\mathbf{B}, \widehat{\omega}) = \inf_{\mathbf{x} \in \overline{\widehat{\omega}}} E(\mathbf{B}, \widehat{\Pi}_{\mathbf{x}})$$

Proposition

$$\lim_{\varepsilon \to 0} E_{\mathrm{ess}}(\mathbf{B}, \mathcal{C}_{\omega_\varepsilon}) = \mathscr{E}(\mathbf{B}, \widehat{\omega}) > 0$$

Since

$$E_n(\mathbf{B}, \omega_\varepsilon) \leq (4n-1) e(\mathbf{B}, \omega)\, \varepsilon \to 0 \textit{ as } \varepsilon \to 0$$

there exists an eigenvalue below the essential spectrum for ε small enough

$+$ corner concentration

cf. [BN-Dauge-Popoff-Raymond 15]