Non self adjoint spectral problems related to a time dependent linearized model in superconductivity.
Part B.
(after Almog-Helffer-Pan, Henry, Grebenkov-Henry-Helffer)

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This work was initially motivated by a paper of Yaniv Almog at Siam J. Math. Appl. [Alm2]. The main goal is to show how some non self-adjoint operators appear in a specific problem appearing in superconductivity and to prove the non emptyness of the spectrum. These results are obtained together with Y. Almog and X. Pan [AlmHelPan1, AlmHelPan2, AlmHelPan3]. Two more papers have been written Almog-Helffer (cmp 2014), Almog-Helffer-Pan (Submitted)

The model in superconductivity

Consider a superconductor placed in an applied magnetic field and submitted to an electric current through the sample. It is usually said that if the applied magnetic field is sufficiently high, or if the electric current is strong, then the sample is in a normal state. We are interested in analyzing the joint effect of the applied field and the current on the stability of the normal state.
To be more precise, let us consider a two-dimensional superconducting sample capturing the entire xy plane. We could have assumed also that a magnetic field of magnitude $\mathcal{H}^e$ is applied perpendicularly to the sample but we take it here such that $\text{curl } \mathcal{H}_e = 0$. Denote the Ginzburg-Landau parameter of the superconductor by κ and the normal conductivity of the sample by σ.

The physical problem is posed in a domain Ω with specific boundary conditions.
This will be discussed in Part C.
In this part, as in Part A, we continue to work in $2D$ and only analyze limiting situations where the domains possibly after a blowing argument become the whole space (or the half-space).

Then the time-dependent Ginzburg-Landau system (also known as the Gorkov-Eliashberg equations) is in $(0, T) \times \mathbb{R}^2$:

$$\begin{cases} \partial_t \psi + i\,\kappa\,\Phi\psi = \nabla^2_{\kappa\mathbf{A}}\psi + \kappa^2\,(1-|\psi|^2)\psi\,, \\ \kappa^2\ \text{curl}\ ^2\mathbf{A} + \sigma\,(\partial_{\mathbf{t}}\mathbf{A} + \nabla\mathbf{\Phi}) = \kappa\,\mathrm{Im}\,(\bar{\psi}\nabla_{\kappa\mathbf{A}}\psi)\,, \end{cases} \tag{1}$$

where ψ is the order parameter, $\mathbf{A}$ is the magnetic potential, Φ is the electric potential, and $(\psi, \mathbf{A}, \mathbf{\Phi})$ also satisfies an initial condition at $t = 0$.

Stationary normal solutions

From (66) we see that if $(0, \mathbf{A}, \mathbf{\Phi})$ is a time-independent normal state solution then $(\mathbf{A}, \mathbf{\Phi})$ satisfies the equality

$$\kappa^2 \text{ curl }^2\mathbf{A} + \sigma\nabla\mathbf{\Phi} = \mathbf{0}\,, \;\; \text{div } \mathbf{A} = \mathbf{0} \quad \text{in } \mathbb{R}^2_{\mathbf{xy}}\,. \tag{2}$$

This could be rewritten as the property that

$$\kappa^2(\text{ curl } \mathbf{A}) + \mathbf{i}\sigma\mathbf{\Phi}\,,$$

is an holomorphic function of $z = x + iy$.

Results by Almog-Helffer-Pan: Φ affine

(66) has the following stationary normal state solution

$$\mathbf{A} = \frac{\mathbf{1}}{\mathbf{2J}}(\mathbf{Jx} + \mathbf{h})^{\mathbf{2}}\hat{\mathbf{i}}_{\mathbf{y}}, \quad \boldsymbol{\Phi} = \frac{\kappa^{\mathbf{2}}\mathbf{J}}{\sigma}\mathbf{y}\,. \tag{3}$$

Note that

$$\text{curl } \mathbf{A} = (\mathbf{Jx} + \mathbf{h})\,,$$

that is, the induced magnetic field equals the sum of the applied magnetic field h and the magnetic field produced by the electric current Jx .

For this normal state solution, the linearization of (66) with respect to the order parameter is

$$\partial_t\psi + \frac{i\kappa^3 Jy}{\sigma}\psi = \Delta\psi - \frac{i\kappa}{J}(Jx+h)^2\partial_y\psi - (\frac{\kappa}{2J})^2(Jx+h)^4\psi + \kappa^2\psi\,. \tag{4}$$

Applying the transformation $x \to x - h/J$ and $\kappa = 1$ for simplification the time-dependent linearized Ginzburg-Landau equation takes the form

$$\frac{\partial\psi}{\partial t} + i\frac{J}{\sigma}y\psi = \Delta\psi - iJx^2\frac{\partial\psi}{\partial y} - \Big(\frac{1}{4}J^2x^4 - 1\Big)\psi\,. \tag{5}$$

Rescaling x and t by applying

$$t \to J^{2/3} t \; ; \; (x,y) \to J^{1/3}(x,y)\,, \tag{6}$$

yields

$$\partial_t u = -(\mathcal{A}_{0,c} - \lambda) u\,, \tag{7}$$

where

$$\mathcal{A}_{0,c} := D_x^2 + (D_y + \frac{1}{2}x^2)^2 + icy\,, \tag{8}$$

and

$$c = 1/\sigma \; ; \; \lambda = J^{-2/3} \; ; \; u(x,y,t) = \psi(J^{-1/3}x, J^{-1/3}y, J^{-2/3}t)\,.$$

Our main problem will be to analyze the long time property of the attached semi-group.
We recall that

$$\mathcal{A}_{0,c} := D_x^2 + (D_y + \frac{1}{2}x^2)^2 + icy \,,$$

Theorem: Decay for the Airy-Montgomery operator

If $c \neq 0$, $\mathcal{A} = \overline{\mathcal{A}_{0,c}}$ has compact resolvent, empty spectrum, and there exist C, t_0 such that, for $t \geq t_0$,

$$\| \exp(-t\mathcal{A}) \| \leq \exp\big(-\frac{2\sqrt{2|c|}}{3} t^{3/2} + Ct^{3/4}\big). \tag{9}$$

We have also a lower bound using a quasimode construction.

Around quantitative Gearhardt-Prüss semi-group

The details for this part can be found in my book: Spectral theory and its applications in Cambridge University Press (2013). The new results (in comparison with the standard books in Semi-group theory) concern the so called quantitative Gearhardt-Prüss theorem and have been obtained in collaboration with J. Sjöstrand (2010).

Definition: Accretive operators

Let A be an unbounded operator in $\mathcal{H}$ with domain $D(A)$. We say that A is accretive if

$$\mathrm{Re}\,\langle Ax\,,\,x\rangle_{\mathcal{H}} \geq 0\,,\,\forall x \in D(A)\,. \tag{10}$$

Proposition

Let A be an accretive operator with a domain $D(A)$ dense in $\mathcal{H}$. Then A is closable and its closed extension $\overline{A}$ is accretive.

Once we have this proposition it is natural to ask for uniqueness of this accretive extension. This leads to the following definition.

Definition

An accretive operator A is maximally accretive if it does not exist an accretive extension $\tilde{A}$ with strict inclusion of $D(A)$ in $D(\tilde{A})$.

The following criterion, which extends the standard criterion of essential self-adjointness is the most suitable.

Theorem

For an accretive operator A, the following conditions are equivalent

1. $\overline{A}$ is maximally accretive.
2. There exists $\lambda_0 > 0$ such that $A^* + \lambda_0 I$ is injective.
3. There exists $\lambda_1 > 0$ such that the range of $A + \lambda_1 I$ is dense in $\mathcal{H}$.

Introduction to semigroups and around the Hille-Yosida theorem

Let us start with simple natural definitions. If $\mathcal{H}$ is a Hilbert space, a one-parameter semigroup of operators on $\mathcal{H}$ is a family of operators indexed on the nonnegative real numbers $\{T(t)\}_{t\in[0,+\infty[}$ such that

$$T(0) = I\,,\; T(s+t) = T(s)\circ T(t)\,, \forall t,s \geq 0\,. \tag{11}$$

Definition

The semigroup is said to be strongly continuous if, for any $x \in \mathcal{H}$, the mapping $t \mapsto T(t)x$ is continuous from $[0,+\infty[$ into $\mathcal{H}$.

Using the semigroup property and the Banach-Steinhaus theorem, it follows easily:

Proposition

If $T(t)$ is a strongly continuous semigroup, then there exists $M \geq 1$ and $\omega_0 \in \mathbb{R}$ such that $T(t)$ has the property

$$P(M, \omega_0): \quad \|T(t)\| \leq M e^{\omega_0 t}, \ t \geq 0. \tag{12}$$

The infinitesimal generator of a one-parameter semigroup T is an unbounded operator A, where $D(A)$ is the set of the $x \in \mathcal{H}$ such that $h^{-1}(T(h)x - x)$ has a limit as h approaches 0 from the right. The value of Ax is the value of the above limit.

Basics on semigroup theory

Existence and uniqueness

Suppose that A generates a strongly continuous semigroup $T(t)$ on $\mathcal{H}$. Then, for any $u_0 \in D(A)$, $u(t) := T(t)u_0$ satisfies

1. $u \in C^0([0,+\infty[, D(A)) \cap C^1([0,+\infty[, \mathcal{H})$;
2. $u'(t) = Au(t)\,,\ u(0) = u_0\,;$
3. $Au(t) = T(t)Au_0\,.$

Moreover the solution of the last equation is unique.

Corollary

Under the same assumptions, for any $f \in C^0([0,T[,\mathcal{H})$ and any $u_0 \in D(A)$, the unique solution of

$$u'(t) - Au(t) = f(t)\,, 0 < t < T\,,\; u(0) = u_0\,,$$

is given by

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s)ds\,.$$

The Hille-Yosida theorem provides a necessary and sufficient condition for a closed linear operator A on an Hilbert space to be the infinitesimal generator of a strongly continuous one-parameter semigroup .

Hille Yosida theorem

Let A be a linear operator defined on a linear subspace $D(A)$ of the Hilbert space $\mathcal{H}$, ω a real number and $M > 0$.
Then A generates a strongly continuous semigroup T that satisfies $P(M,\omega)$, if and only if

1. $D(A)$ is dense in $\mathcal{H}$, and
2. every real $\lambda > \omega$ belongs to the resolvent set of A and for such λ and for all positive integers n

$$||(\lambda I - A)^{-n}|| \leq \frac{M}{(\lambda - \omega)^n} . \tag{13}$$

The explicit statement of the Hille-Yosida theorem for contraction semigroups is contained (case $\omega = 0$) in:

Theorem

Let A be a linear operator defined on a linear subspace $D(A)$ of the Hilbert space $\mathcal{H}$. Then A generates a continuous semigroup satisfying $P(1, \omega)$ if and only if

1. $D(A)$ is dense in $\mathcal{H}$
2. every real $\lambda > 0$ belongs to the resolvent set of A and for such λ

$$||(\lambda I - A)^{-1}|| \leq \frac{1}{(\lambda - \omega)} . \tag{14}$$

What one can actually prove is that $P(M,\omega)$ implies that every $z \in \mathbb{C}$, such that $\mathrm{Re}\, z > \omega$ belongs to the resolvent set of A and satisfies, for any $n \geq 1$,

$$||(zI - A)^{-n}|| \leq \frac{M}{(\mathrm{Re}\, z - \omega)^n} \,. \tag{15}$$

It is then natural to ask for a converse using the previous inequality (with $n = 1$), this is true if one write the inequality for $z \in \mathbb{C}$.

The Gearhart-Prüss theorem

We next state:

Gearhart-Prüss (GP) Theorem

Let A be a closed operator with dense domain $D(A)$ and $\omega \in \mathbb{R}$. Assume that $\|(z - A)^{-1}\|$ is uniformly bounded in the half-plane $\mathrm{Re}\, z \geq \omega$. Then there exists a constant $M > 0$ such that $P(M, \omega)$ holds.

Hence, the control of the decay of the associated semi-group depends on the control of the norm of the resolvent in some half-plane.

Reformulation in terms of ϵ-spectra

For a given accretive closed operator $\mathcal{A}$, we introduce for any $\epsilon > 0$,

$$\widehat{\alpha}_\epsilon(\mathcal{A}) = \inf_{z \in \Sigma_\epsilon(\mathcal{A})} \mathrm{Re}\, z \,. \tag{16}$$

Here $\Sigma_\epsilon(\mathcal{A})$ is the ϵ-pseudospectrum of $\mathcal{A}$:

$$\Sigma_\epsilon(\mathcal{A}) ;= \{z \,,\, ||(z - \mathcal{A})^{-1}|| > \frac{1}{\epsilon} \} \,,$$

with the convention that $||(z - \mathcal{A})^{-1}|| = +\infty$ if $z \in \sigma(\mathcal{A})$.
We also define

$$\widehat{\omega}_0(\mathcal{A}) = \lim_{t \to +\infty} \frac{1}{t} \log || \exp -t\mathcal{A}|| \,. \tag{17}$$

Gearhart-Prüss theorem reformulated

Let $\mathcal{A}$ be a densely defined closed operator in an Hilbert space X such that $-\mathcal{A}$ generates a contraction semigroup and let $\widehat{\alpha}_\epsilon(\mathcal{A})$ and $\widehat{\omega}_0(\mathcal{A})$ denote the ϵ-pseudospectral abcissa and the growth bound of $\mathcal{A}$ respectively. Then

$$\lim_{\epsilon \to 0} \widehat{\alpha}_\epsilon(\mathcal{A}) = -\widehat{\omega}_0(\mathcal{A})\,. \tag{18}$$

This version is interesting because it reduces the question of the decay, which is basic in the question of the stability to an analysis of the ϵ-spectra of the operator for ϵ small.

Definition

$$\omega_0 = \inf\{\omega \in \mathbb{R}\, \{z \in \mathbb{C}; \mathrm{Re}\, z > \omega\} \subset \rho(A) \\ \text{and } \sup_{\mathrm{Re}\, z > \omega} \|(z-A)^{-1}\| < \infty\}\,, \tag{19}$$

and for $\omega > \omega_0$, let $r(\omega)$ be defined by

$$\frac{1}{r(\omega)} = \sup_{\mathrm{Re}\, z > \omega} \|(z-A)^{-1}\|. \tag{20}$$

Then $r(\omega)$ is an increasing function of ω, $\omega - r(\omega) \geq \omega_0$ and for $\omega' \in [\omega - r(\omega), \omega]$ we have $r(\omega') \geq r(\omega) - (\omega - \omega')$.
Moreover, if $\omega_0 > -\infty$, then $r(\omega) \to 0$ when $\omega \searrow \omega_0$.

The main result is due to Helffer-Sjöstrand (2010) (see also Mouhot):

Quantitative Gearhart-Prüss (QGP)

Under the assumptions of Theorem GP, let $m(t)$ be a continuous positive function such that:

$$m(t) \geq \|S(t)\| .$$

Then for all triple $t, a, \widetilde{a} > 0$, such that $t = a + \widetilde{a}$, we have

$$\|S(t)\|_{\mathcal{L}(\mathcal{H})} \leq \frac{e^{\omega t}}{r(\omega)\|\frac{1}{m}\|_{e^{-\omega\cdot}L^2([0,a])}\|\frac{1}{m}\|_{e^{-\omega\cdot}L^2([0,\widetilde{a}])}} . \tag{21}$$

Above $\|f\|_{e^{-\omega\cdot}L^2([0,a])} = \|e^{\omega\cdot}f(\cdot)\|_{L^2([0,a])}$.

Applications

The theorem is based on two assumptions: the existence of some initial control by $m(t)$ and the additional information on the resolvent. For example, Hille-Yosida's theorem shows that we can take $m(t) = \widehat{M} \exp \widehat{\omega} t$, for some $\widehat{\omega} \geq \omega$, with actually $\widehat{M} = 1$. We apply Theorem QGP with this $m(t)$ and $a = \tilde{a} = \frac{t}{2}$. The term appearing in the denominator of (21) becomes

$$\|\frac{1}{m}\|_{e^{-\omega \cdot} L^2([0,a])} \|\frac{1}{m}\|_{e^{-\omega \cdot} L^2([0,\tilde{a}])} = \frac{1}{2} \widehat{M}^{-2} t \,, \tag{22}$$

if $\widehat{\omega} = \omega$, and

$$= \frac{1}{2\widehat{M}^2(\widehat{\omega} - \omega)} \left[1 - \exp((\omega - \widehat{\omega})t)\right] \,, \tag{23}$$

if $\widehat{\omega} > \omega$.

Proposition

Let $T(t)$ be a continuous semigroup such that $P(\widehat{M},\widehat{\omega})$ is satisfied for some pair $(\widehat{M},\widehat{\omega})$ and such that $r(\omega) > 0$ for some $\omega < \widehat{\omega}$.
Then:

$$||T(t)|| \leq \widehat{M}\left(1+\frac{2\widehat{M}(\widehat{\omega}-\omega)}{r(\omega)}\right)\exp \omega t\,. \tag{24}$$

The case $\omega = \widehat{\omega} = 0$

If we assume the control of the norm of the resolvent on $\mathrm{Re}\, z \geq 0$, we get

$$||T(t)|| \leq \frac{2\widehat{M}}{r(0)t}\,,$$

Using the semigroup property shows that

$$||T(t)|| \leq \left(\frac{2\widehat{M}N}{r(0)t}\right)^N\,,$$

for any $N \geq 1$. Optimizing over N permits to deduce an exponential decay of $S(t)$ at ∞.

Return to equilibrium

We also have the following variant of the main result that can be useful in problems of return to equilibrium.

GP theorem and spectral gap

Let $\widetilde{\omega} < \omega$ and assume that A has no spectrum on the line $\mathrm{Re}\, z = \widetilde{\omega}$ and that the spectrum of A in the half-plane $\mathrm{Re}\, z > \widetilde{\omega}$ is compact (and included in the strip $\widetilde{\omega} < \mathrm{Re}\, z < \omega$). Assume that $\|(z-A)^{-1}\|$ is uniformly bounded on $\{z \in \mathbb{C};\, \mathrm{Re}\, z \geq \widetilde{\omega}\} \setminus U$, for any neighborhood U of $\sigma_+(A) := \{z \in \sigma(A);\, \mathrm{Re}\, z > \widetilde{\omega}\}$ and define $r(\widetilde{\omega})$ by

$$\frac{1}{r(\widetilde{\omega})} = \sup_{\mathrm{Re}\, z = \widetilde{\omega}} \|(z-A)^{-1}\|.$$

Then for every $t > 0$,

$$S(t) = S(t)\Pi_+ + R(t) = S(t)\Pi_+ + S(t)(1 - \Pi_+),$$

where for all $a, \widetilde{a} > 0$ with $a + \widetilde{a} = t$,

$$\|R(t)\| \leq \frac{e^{\widetilde{\omega} t}}{r(\widetilde{\omega}) \|\frac{1}{m}\|_{e^{-\widetilde{\omega}\cdot} L^2([0,a])} \|\frac{1}{m}\|_{e^{-\widetilde{\omega}\cdot} L^2([0,\widetilde{a}])}} \|I - \Pi_+\|. \tag{25}$$

Here Π_+ denotes the spectral projection associated with $\sigma_+(A)$:

$$\Pi_+ = \frac{1}{2\pi i}\int_{\partial V} (z-A)^{-1} dz\,,$$

where V is any compact neighborhood of $\sigma_+(A)$ with C^1 boundary, disjoint from $\sigma(A) \setminus \sigma_+(A)$.

Coming back to the analysis of models

After this reminder in semi-group theory, we come back to the analysis of models but this time with boundary.

Reminder on the complex Airy operator

Here we recall relatively basic facts coming from ... Martinet, Almog, Bordeaux-Montrieux, Helffer, Henry, Almog-Helffer, Almog-Grebenkov-Henry ... and discuss new questions concerning estimates on the resolvent.
This will be a way to understand the more complicate situation of the Montgomery-Airy operator

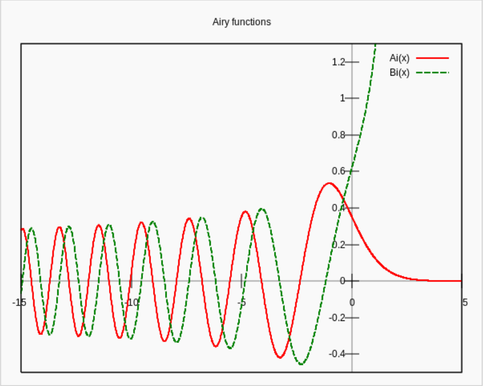
$$D_x^2 + (D_y - \frac{x^2}{2})^2 + icy\,.$$

The Airy solution

This is a specific solution $Ai(x)$ of

$$D_x^2 + xu = 0\,,$$

with the property that u tends to 0 at $+\infty$ (+ some normalization at 0.



La fonction Ai est en rouge et Bi en vert.

The complex Airy operator on the line

Remark: The x here corresponds to the y in the previous slide.
This operator can be defined as the closed extension $\mathcal{A}^+$ of the differential operator $\mathcal{A}_0^+ := D_x^2 + i\,x$ on $C_0^\infty(\mathbb{R})$. We observe that $\mathcal{A}^+ = (\mathcal{A}_0^-)^*$ with $\mathcal{A}_0^- := D_x^2 - i\,x$ and that its domain is

$$D(\mathcal{A}^+) = \{u \in H^2(\mathbb{R}),\, x\,u \in L^2(\mathbb{R})\}\,.$$

In particular $\mathcal{A}^+$ has compact resolvent. It is also easy to see that $-\mathcal{A}^+$ is the generator of a semi-group S_t of contraction,

$$S_t = \exp(-t\mathcal{A}^+)\,. \tag{26}$$

Hence all the results of the theory of semi-groups can be applied (see for example the book by Davies).
In particular, we have, for $\mathrm{Re}\,\lambda < 0$,

$$||(\mathcal{A}^+ - \lambda)^{-1}|| \leq \frac{1}{|\mathrm{Re}\,\lambda|}\,. \tag{27}$$

A very special property of this operator is that, for any $a \in \mathbb{R}$,

$$T_a\,\mathcal{A}^+ = (\mathcal{A}^+ - ia)\,T_a\,, \tag{28}$$

where T_a is the translation operator $(T_a u)(x) = u(x-a)$.

As immediate consequence, we obtain that the spectrum is empty and that the resolvent of $\mathcal{A}^+$,

$$\mathcal{G}_0^+(\lambda) = (\mathcal{A}^+ - \lambda)^{-1}$$

which is defined for any $\lambda \in \mathbb{C}$, satisfies

$$||(\mathcal{A}^+ - \lambda)^{-1}|| = ||(\mathcal{A}^+ - \mathrm{Re}\,\lambda)^{-1}|| \,. \tag{29}$$

The most interesting property is the control of the resolvent for $\mathrm{Re}\,\lambda \geq 0$.

Proposition (W. Bordeaux-Montrieux)

As $\mathrm{Re}\,\lambda \to +\infty$, we have

$$||\mathcal{G}_0^+(\lambda)|| \sim \sqrt{\frac{\pi}{2}} (\mathrm{Re}\,\lambda)^{-\frac{1}{4}} \exp\left(\frac{4}{3} (\mathrm{Re}\,\lambda)^{\frac{3}{2}}\right) \,. \tag{30}$$

This improves a previous result on the Hilbert-Schmidt norm by J. Martinet (see also the book Spectral Theory and its applications by B. Helffer (last three chapters))

$$\|\mathcal{G}_0^+(\lambda)\|_{HS} = \|\mathcal{G}_0^+(\mathrm{Re}\,\lambda)\|_{HS}\,, \tag{31}$$

and

$$\|\mathcal{G}_0^+(\lambda)\|_{HS} \sim \sqrt{\pi}\,(\mathrm{Re}\,\lambda)^{-\frac{1}{4}} \exp\left(\frac{4}{3}(\mathrm{Re}\,\lambda)^{\frac{3}{2}}\right)\,, \tag{32}$$

as $\mathrm{Re}\,\lambda \to +\infty$.
Here $\|\cdot\|_{HS}$ is the Hilbert-Schmidt norm.
This is consistent with the well-known translation invariance properties of the operator $\mathcal{A}_0$.

The proof of the (rather standard) upper bound is based on the direct analysis of the semigroup in the Fourier representation. We note indeed that

$$\mathcal{F}(D_x^2 + i\,x)\mathcal{F}^{-1} = \xi^2 - \frac{d}{d\xi}\,. \tag{33}$$

Using the characteristic method for the evolution problem in the Fourier side, we easily get:

$$\mathcal{F} S_t \mathcal{F}^{-1} v = \exp(-\xi^2 t - \xi t^2 - \frac{t^3}{3}) v(\xi + t)\,, \tag{34}$$

and this implies immediately

$$||S_t|| = \exp \max_{\xi}(-\xi^2 t - \xi t^2 - \frac{t^3}{3}) = \exp(-\frac{t^3}{12})\,. \tag{35}$$

Note that this decay implies by Gearhart-Prüss theorem that the spectrum is empty.
Then one can get an estimate of the resolvent by using, for $\lambda \in \mathbb{C}$, the formula

$$(\mathcal{A} - \lambda)^{-1} = \int_0^{+\infty} \exp -t(\mathcal{A} - \lambda)\, dt\,. \tag{36}$$

For a closed accretive operator, (36) is standard when $\mathrm{Re}\,\lambda < 0$, but estimate (35) on S_t gives immediately an holomorphic extension of the right hand side to the whole space giving for $\lambda > 0$ the estimate

$$||(\mathcal{A} - \lambda)^{-1}|| \leq \int_0^{+\infty} \exp(\lambda t - \frac{t^3}{12})\, dt\,. \tag{37}$$

The asymptotic behavior as $\lambda \to +\infty$ of this integral is immediately obtained by using the Laplace method and the dilation $t = \lambda^{\frac{1}{2}} s$ in the integral.

In our case, everything can be computed explicitly. We observe that:

$$\int_0^{+\infty} \exp(-\xi^2 t - \xi t^2 - \frac{t^3}{3}) e^{\lambda t} v(\xi + t) dt$$
$$= \int_\xi^{+\infty} \exp\left(\frac{\xi^3 - s^3}{3} + \lambda(s - \xi)\right) v(s)\, ds\,,$$

which gives effectively and expression of $(\mathcal{A} - \lambda)^{-1}$.

Quasi-modes

The proof of the lower bound is obtained by constructing quasimodes for the operator $(\mathcal{A} - \lambda)$ in its Fourier representation. We observe (assuming $\lambda > 0$), that

$$\xi \mapsto u(\xi;\lambda) := \exp\left(-\frac{\xi^3}{3} + \lambda\xi - \frac{2}{3}\lambda^{\frac{3}{2}}\right) \quad (38)$$

is a solution of

$$(\frac{d}{d\xi} + \xi^2 - \lambda)u(\xi;\lambda) = 0\,. \quad (39)$$

Multiplying $u(\cdot;\lambda)$ by a cut-off function χ_λ with support in $]-\sqrt{\lambda}, +\infty[$ and $\chi_\lambda = 1$ on $]-\sqrt{\lambda}+1, +\infty[$, we obtain a very good quasimode, concentrated as $\lambda \to +\infty$, around $\sqrt{\lambda}$, with an error term giving almost[1] the announced lower bound for the resolvent.

[1] One should indeed improve the cut-off for getting an optimal result.

Coming back to the Montgomery-(Complex)Airy operator

The proof for the Montgomery-(complex)Airy operator is much more difficult but some of the arguments are the same (for example the translation argument). The proof of compactness of the resolvent is reminiscent of Kohn's proof of the hypoellipticity of the Hörmander's operators in the form $\sum_j X_j^2 + X_0$ which reappear first in Helffer-Mohamed and later in the analysis of Fokker-Planck operator (Helffer-Mohamed, Herau-Nier, Eckmann-Hairer, Villani...).

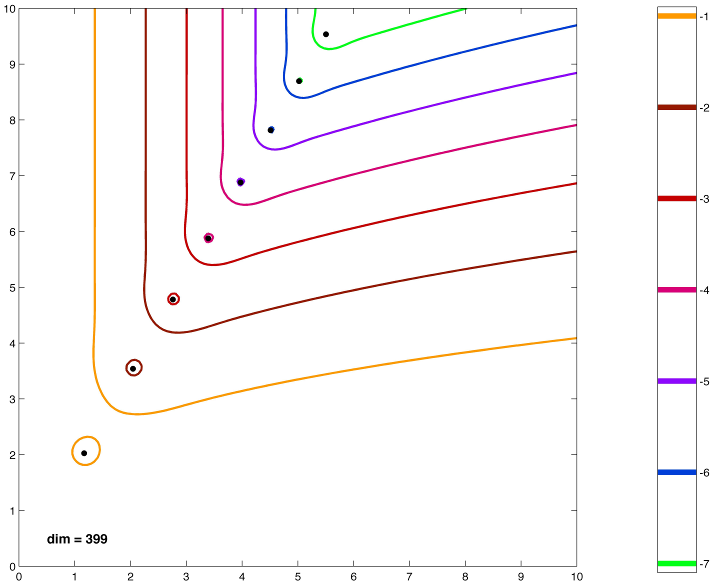
The complex Airy operator on the half-line: Dirichlet case

It is easy to define the Dirichlet realization of $D_x^2 \pm ix$ on $\mathbb{R}_+$ (the analysis on the negative semi-axis is similar). One can use for example the Lax Milgram theorem and take as form domain

$$V^D = \{u \in H_0^1(\mathbb{R}_+)\,,\, x^{\frac{1}{2}} u \in L^2_+\}\,.$$

It can also be shown, that the domain is

$$\mathcal{D}^D := \{u \in V^D\,,\, u \in H^2_+\,,\, xu \in L^2_+\}\,.$$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This implies the

Proposition

The resolvent $\mathcal{G}^{\pm,D}(\lambda) := (\mathcal{A}^{\pm,D} - \lambda)^{-1}$ is in the Schatten class C^p for any $p > \frac{3}{2}$, where $\mathcal{A}^{\pm,D} = D_x^2 \pm ix$ and the superscript D refers to the Dirichlet case.

More precisely we provide the distribution kernel $\mathcal{G}^{-,D}(x,y;\lambda)$ of the resolvent for the complex Airy operator $D_x^2 - ix$ on the positive semi-axis with Dirichlet boundary condition at the origin (the results for $\mathcal{G}^{+,D}(x,y;\lambda)$ are similar). Matching the boundary conditions, one gets

$$\mathcal{G}^{-,D}(x,y;\lambda) = \begin{cases} 2\pi \frac{\mathrm{Ai}(e^{-i\alpha}w_y)}{\mathrm{Ai}(e^{-i\alpha}w_0)} \big[\mathrm{Ai}(e^{i\alpha}w_x)\mathrm{Ai}(e^{-i\alpha}w_0) \\ \qquad -\mathrm{Ai}(e^{-i\alpha}w_x)\mathrm{Ai}(e^{i\alpha}w_0)\big] \quad (0 < x < y), \\ 2\pi \frac{\mathrm{Ai}(e^{-i\alpha}w_x)}{\mathrm{Ai}(e^{-i\alpha}w_0)} \big[\mathrm{Ai}(e^{i\alpha}w_y)\mathrm{Ai}(e^{-i\alpha}w_0) \\ \qquad -\mathrm{Ai}(e^{-i\alpha}w_y)\mathrm{Ai}(e^{i\alpha}w_0)\big] \quad (x > y), \end{cases} \tag{40}$$

where $\mathrm{Ai}(z)$ is the Airy function, $w_x = ix + \lambda$, and $\alpha = 2\pi/3$.

The above expression can also be written as

$$\mathcal{G}^{-,D}(x,y;\lambda) = \mathcal{G}_0^{-}(x,y;\lambda) + \mathcal{G}_1^{-,D}(x,y;\lambda), \tag{41}$$

where $\mathcal{G}_0^{-}(x,y;\lambda)$ is the resolvent for the Airy operator $D_x^2 - ix$ on the whole line,

$$\mathcal{G}_0^{-}(x,y;\lambda) = \begin{cases} 2\pi \mathrm{Ai}(e^{i\alpha} w_x)\mathrm{Ai}(e^{-i\alpha} w_y) & (x < y), \\ 2\pi \mathrm{Ai}(e^{-i\alpha} w_x)\mathrm{Ai}(e^{i\alpha} w_y) & (x > y), \end{cases} \tag{42}$$

and

$$\mathcal{G}_1^{-,D}(x,y;\lambda) = -2\pi \frac{\mathrm{Ai}(e^{i\alpha}\lambda)}{\mathrm{Ai}(e^{-i\alpha}\lambda)} \mathrm{Ai}\big(e^{-i\alpha}(ix+\lambda)\big)\, \mathrm{Ai}\big(e^{-i\alpha}(iy+\lambda)\big). \tag{43}$$

The resolvent is compact. The poles of the resolvent are determined by the zeros of $\mathrm{Ai}(e^{-i\alpha}\lambda)$, i.e., $\lambda_n = e^{i\alpha} a_n$, where the a_n are zeros of the Airy function: $\mathrm{Ai}(a_n) = 0$. The eigenvalues have multiplicity 1 (no Jordan block).

We have, as a consequence of the analysis of the numerical range of the operator,

Proposition

$$||\mathcal{G}^{\pm,D}(\lambda)|| \leq \frac{1}{|\mathrm{Re}\,\lambda|}\,, \text{ if } \mathrm{Re}\,\lambda < 0\,; \tag{44}$$

and

$$||\mathcal{G}^{\pm,D}(\lambda)|| \leq \frac{1}{|\mathrm{Im}\,\lambda|}\,, \text{ if } \mp \mathrm{Im}\,\lambda > 0\,. \tag{45}$$

Finally, if we want to apply Gearhard-Prüss theorem, we should also prove that for $\mathrm{Re}\,\lambda$ in a compact interval, then the resolvent is bounded as $\pm \mathrm{Im}\,\lambda \to +\infty$. See the pseudo-spectra figure.

This proposition together with the Phragmen-Lindelöf principle (see Agmon or Dunford-Schwartz) leads to:

Proposition

The space generated by the eigenfunctions of the Dirichlet realization for $D_x^2 \pm ix$ is dense in L^2_+.

Note that it is proven by Henry (2013) that one can not find a Riesz basis of eigenfunctions.

At the boundary of the numerical range of the operator, it is interesting to analyze the behavior of the resolvent. Numerical computations lead to

Conjecture

When λ is real

$$\lim_{\lambda \to +\infty} ||\mathcal{G}^{\pm,D}(\lambda)|| = 0\,. \tag{46}$$

The complex Airy operator on the half-line: Neumann case

Similarly, we can look at the Neumann realization of $D_x^2 - ix$ on $\mathbb{R}_+$. One can use for example the Lax-Milgram theorem and take as form domain

$$V^N = \{u \in H^1_+\,,\, x^{\frac{1}{2}} u \in L^2_+\}.$$

The Neumann condition appears when writing the domain of the operator.

More explicitly, the resolvent of the Neumann problem for $D_x^2 - ix$ is obtained as

$$\mathcal{G}^{-,N}(x,y;\lambda) = \mathcal{G}_0^{-}(x,y;\lambda) + \mathcal{G}_1^{-,N}(x,y;\lambda)\,,$$

for $(x,y) \in \mathbb{R}_+^2$, where $\mathcal{G}_1^{-,N}(x,y;\lambda)$ is given by the following expression:

$$\mathcal{G}_1^{-,N}(x,y;\lambda) = -2\pi \frac{e^{i\alpha}\mathrm{Ai}'(e^{i\alpha}\lambda)}{e^{-i\alpha}\mathrm{Ai}'(e^{-i\alpha}\lambda)} \mathrm{Ai}\big(e^{-i\alpha}(ix+\lambda)\big)\,\mathrm{Ai}\big(e^{-i\alpha}(iy+\lambda)\big)\,. \tag{47}$$

The resolvent is compact. The poles of the resolvent are determined by zeros of $\mathrm{Ai}'(e^{-i\alpha}\lambda)$, i.e., $\lambda_n = e^{i\alpha} a_n'$, where a_n' are zeros of the derivative of the Airy function: $\mathrm{Ai}'(a_n') = 0$. The eigenvalues have multiplicity 1 (no Jordan block).
We have, as a consequence of the analysis of the numerical range of the operator,

Proposition

$$||\mathcal{G}^{+,N}(\lambda)|| \leq \frac{1}{|\mathrm{Re}\,\lambda|}\,, \text{ if } \mathrm{Re}\,\lambda < 0\,; \tag{48}$$

and

$$||\mathcal{G}^{+,N}(\lambda)|| \leq \frac{1}{|\mathrm{Im}\,\lambda|}\,, \text{ if } \mp \mathrm{Im}\,\lambda > 0\,. \tag{49}$$

This proposition together with the Phragmen-Lindelöf principle implies the completeness of the eigenfunctions:

Proposition

The space generated by the eigenfunctions of the Neumann problem for $D_x^2 \pm ix$ is dense in L^2_+.

At the boundary of the numerical range of the operator, it is interesting to analyze the behavior of the resolvent. Numerical computations lead to

Conjecture

When λ is real

$$\lim_{\lambda \to +\infty} ||\mathcal{G}^{\pm,N}(\lambda)|| = 0\,. \qquad (50)$$

Remark

Using a Wronskian argument, we have

$$\mathcal{G}^{-,D}(x,y;\lambda) - \mathcal{G}^{-,N}(x,y;\lambda) = -ie^{i\alpha}\frac{\mathrm{Ai}(e^{-i\alpha}w_x)\mathrm{Ai}(e^{-i\alpha}w_y)}{\mathrm{Ai}(e^{-i\alpha}\lambda)\mathrm{Ai}'(e^{-i\alpha}\lambda)}\,. \tag{51}$$

Computations which will be done later show that for $\lambda \geq 1$

$$||\mathcal{G}^{-,D}(\lambda) - \mathcal{G}^{-,N}(\lambda)||_{HS} \leq C\,|\lambda|^{-\frac{1}{4}}\,. \tag{52}$$

Hence the conjectures for Dirichlet and Neumann are equivalent. We expect actually a decay in $\lambda^{-\frac{1}{4}}(\log\lambda)^{\frac{1}{2}}$.

The case of the half-space

As the operator $\mathcal{A}_{0,c}$ is defined on $C_0^\infty(\mathbb{R}^2_+)$, we use Friedrichs construction starting from sesquilinear form :
$(u, v) \longmapsto$

$$\tilde{a}(u, v) = \langle D_x u \, , \, D_x v \rangle + \langle (D_y + \frac{x^2}{2}) u \, , \, (D_y + \frac{x^2}{2}) v \rangle + ic \int_0^{+\infty} y u \bar{v} \, dx dy \, , \tag{53}$$

Once the definition of the extended operator $\mathcal{A}_c^+$ has been formulated, we may write

$$\mathcal{A}_c^+ = D_x^2 + (D_y + \frac{1}{2}x^2)^2 + icy\,. \tag{54}$$

Note that $\mathcal{A}_c^+$ is not self-adjoint. Furthermore, we have that

$$(\mathcal{A}_c^+)^* = \mathcal{A}_{-c}^+\,.$$

In the present contribution we analyze the spectrum of $\mathcal{A}_c^+$, denoted by $\sigma(\mathcal{A}_c^+)$, and the associated semi-group $\exp -t\mathcal{A}_c^+$.

As in the case of the whole space, we can easily prove the following :

Proposition

For any $c > 0$, $\mathcal{A}_c^+$ has compact resolvent. Moreover, if $E_0(\omega)$ denotes the ground state energy of the Montgomery operator

$$\mathcal{M}_\omega := -\frac{d^2}{dx^2} + \left(\frac{x^2}{2} + \omega\right)^2,$$

and if

$$\lambda_0^M = \inf_{\omega \in \mathbb{R}} \lambda^M(\omega) = \lambda^M(\omega^*)\,, \tag{55}$$

then

$$\sigma(\mathcal{A}_c^+) \subset \{\lambda \in \mathbb{C}\,,\, \mathrm{Re}\,\lambda \geq \lambda_0^M\}. \tag{56}$$

As mentioned earlier, our interest is in the effect that the Dirichlet boundary condition has on the spectrum $\sigma(\mathcal{A}_c^+)$ and on the semigroups $\exp -t\mathcal{A}_c^+$. Thus, it is interesting to compare them with the analogous entities for the whole-plane problem.

We have seen, that $\sigma(\mathcal{A})$ is empty and that the decay of the semigroup $\exp(-t\mathcal{A})$ is faster than any exponential rate.
On the other hand, for the half-plane problem we do not expect $\sigma(\mathcal{A}_c^+)$ to be empty. We provide here a proof in the asymptotic regimes $c \to +\infty$ and $c \to 0$.

Theorem

There exists $c_0 \geq 0$ such that for $c \geq c_0$

$$\sigma(\mathcal{A}_c^+) \neq \emptyset.$$

Furthermore, there exists $\mu(c) \in \sigma(\mathcal{A}_c^+)$ which as $c \to +\infty$:

$$\mu(c) \sim c^{2/3} \exp(i\frac{\pi}{3})\alpha_0 + \lambda_1 \exp(-i\frac{\pi}{6})\, c^{-1/3} + \mathcal{O}(c^{-5/6}), \quad (57)$$

with $-\alpha_0$ the rightmost zero point of Airy's function, and λ_1 an eigenvalue of an harmonic oscillator like operator.

$\mu(c)$ is the candidate to be the eigenvalue with smallest real part. One can indeed show that if

$$\mu_{min}(c) = \inf_{z\in\sigma(\mathcal{A}_c^+)} \mathrm{Re}\, z\,. \tag{58}$$

then, for all $c > c_0$ $\mu_{min}(c) \sim \mathrm{Re}\,\mu(c) + \mathcal{O}(c^{-5/6})\,.$

The next result is based on Gearhard-Pruss Theorem and is valid for all $c > 0$.

Theorem

If $\sigma(\mathcal{A}_c^+) \neq \emptyset$, then

$$\lim_{t \to +\infty} -\frac{\log \| \exp(-t\mathcal{A}_c^+)\|}{t} = \mu_{min}(c)\,. \tag{59}$$

The case of c large, Analytic dilation

Instead of dealing with $\sigma(\mathcal{A}_c^+)$, it is more convenient to analyze the spectrum of the operator $\mathcal{P}_\theta$ which is obtained from $\mathcal{A}_c^+$ using a gauge transformation and analytic dilation.
Let $\theta \in \mathbb{C}$. Like for the analysis of resonances, we introduce the dilation operator

$$u \longmapsto (U(\theta)u)(x,y) = e^{-\theta/2}\, u(e^\theta x\,,\, e^{-2\theta} y)\,. \qquad (60)$$

Set then

$$\mathcal{P}_\theta := U(\theta)^{-1}\mathcal{P}\,U(\theta) = e^{2\theta}\,(D_x - yx)^2 - e^{-4\theta}\,\partial_y^2 + ic\,e^{2\theta}\,y\,, \quad (61)$$

For $\theta = -i\frac{\pi}{12}$, we have

$$\mathcal{P}_{-i\frac{\pi}{12}} = e^{i\pi/3}(D_y^2 + cy) + e^{-i\pi/6}(D_x - xy)^2\,.$$

Note that $\mathcal{P}_{-i\frac{\pi}{12}}$ is *not* unitarily equivalent to $\mathcal{A}_c^+$. But analytic dilation facilitates the analysis of the spectrum of $\mathcal{A}_c^+$. We introduce the (small) parameter $\epsilon = \frac{1}{c}$.

We define another operator via the (real) dilation

$$\mathcal{B}_\epsilon := \epsilon^{2/3} e^{i\pi/6} U(-\ln \frac{\epsilon}{6})^{-1} \mathcal{P}_{-i\frac{\pi}{12}} U(-\ln \frac{\epsilon}{6}) \,, \tag{62}$$

where U is defined in (60):

$$\mathcal{B}_\epsilon := \epsilon \big(D_x - xy\big)^2 + i \big(D_y^2 + y\big) \,. \tag{63}$$

Quasimodes

Proposition

There exist $\{u_j(x,y)\}_{j=0}^{\infty} \subset \mathcal{S}\left(\overline{\mathbb{R}^2_+}\right) \cap D(\mathcal{B}_\epsilon)$ with $\|u_0\|_{L^2} = 1$, and $\{\lambda_j\}_{j=0}^{\infty} \subset \mathbb{C}\,, i^{j+1}\lambda_j$ real for all $j \geq 0$ and $-i\lambda_0 = \alpha_0\,,$ is the lowest eigenvalue of the Airy operator in $\mathbb{R}_+$ with homogeneous Dirichlet condition at 0, s.t.

$$\mathcal{B}_\epsilon\left(\sum_{j=0}^{\infty} \epsilon^j u_j(x,y)\right) \sim \left(\sum_{j=0}^{\infty} \epsilon^j \lambda_j\right)\left(\sum_{j=0}^{\infty} \epsilon^j u_j(x,y)\right)\,. \quad (64)$$

We note that if ϵ had been purely imaginary, then (64) would have implied by the spectral theorem the existence of an eigenvalue $\lambda(\epsilon)$ admitting the complete asymptotic expansion

$$\lambda(\epsilon) \sim \sum_{j=0}^{\infty} \epsilon^j \lambda_j \,. \tag{65}$$

Since our interest is when ϵ is real, a more complicated approach has to be adopted in order to prove the existence of an eigenvalue of $\mathcal{B}_\epsilon$.

Other open problems

Do the spectral analysis of the following models:

$$-\Delta_{x,y} + ic(\cos\theta x + \sin\theta y) \text{ in } y > 0\,,$$

and

$$D_x^2 + (D_y + \frac{\sin\theta}{2}x^2 - \cos\theta xy)^2 + ic(\cos\theta x + \sin\theta y) \text{ in } y > 0\,,$$

with Dirichlet boundary condition at $y = 0$.
The first case is analyzed by R. Henry. The second case is open. Note that the question of the definition of the second model was finally clarified by Almog-Helffer (2014).

Back to the initial problem in the case $y > 0$

As conclusion of this part, we have shown that there is a critical J_{crit} for our modelization of the half space problem in the affine case. For $J > J_{crit}$, the solutions are stable and tend to the normal state as $t \to +\infty$.
With the notation of the beginning of Part B, we have

$$J_{crit}^{-\frac{2}{3}} = \mu_{min}(c)\,.$$

We recall that $c = \frac{1}{\sigma}$ where σ was the normal conductivity, κ being assumed to be 1.

We recall that the time-dependent linearized (at the normal solution) Gorkhov-Eliashberg system reads is in $(0,+\infty)\times\mathbb{R}^2_+$:

$$
\begin{aligned}
\partial_t\psi + i\,\kappa\,\Phi\psi &= \nabla^2_{\kappa\mathbf{A}}\psi + \kappa^2\,\psi\,,\\
\psi(0,\cdot) &= \psi_0(\cdot)\,,\\
\psi(t,x,0) &= 0\,,
\end{aligned}
\tag{66}
$$

where $A(x,y,h) = J\,(x-h)\,,\ \Phi(x,y,h) = \frac{\kappa^2 J}{\sigma}\,y$.

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