The Time-Dependent Hartree-Fock-Bogoliubov Equations for Bosons

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SEV-2013-0323, MTM2014-53850, Marie Sklodowska-Curie 660021

February 11, 2016

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Bose-Einstein Condensate

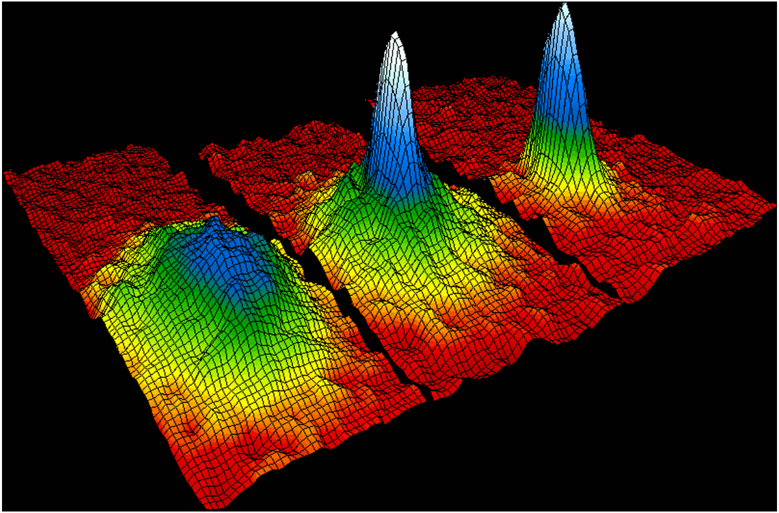

Figure : Velocity-distribution for a gas of rubidium atoms, showing the transition to Bose-Einstein condensation (1995).

Motivation

- Understand the dynamics of a Bose-Einstein condensate.
- In particular the quantum fluctuations with respect to the evolution through the time-dependant Gross-Pitaevskii equation.

Model for Many Bosons

- Hamiltonian on $L^2_s(\mathbb{R}^{dN})$ of the form

$$H_N = \sum_{n=1}^{N} -\Delta_{x_n} + \sum_{1\le n<m\le N} v(x_m - x_n)$$

- An approximation is

$$e^{-itH_N}\phi_0^{\otimes N} \simeq \phi_t^{\otimes N}$$

with ϕ_t solution to the Hartree equation:

$$i\partial_t\phi_t = -\Delta\phi_t + (v * |\phi_t|^2)\phi_t \,.$$

Model for Many Bosons

- Hamiltonian on $\mathcal{F} = \bigoplus_{N\geq 0} L_s^2(\mathbb{R}^{dN})$ of the form

$$\mathbb{H} = \bigoplus_{N\geq 0} H_N = \bigoplus_{N\geq 0} \left(\sum_{n=1}^{N} -\Delta_{x_n} + \sum_{1\leq n<m\leq N} v(x_m - x_n) \right)$$

- An approximation is

$$e^{-it\mathbb{H}} E(\phi_0) = \bigoplus_{N\geq 0} \frac{e^{-\frac{\|\phi_0\|^2}{2}}}{\sqrt{N!}} e^{-itH_N} \phi_0^{\otimes N} \simeq \bigoplus_{N\geq 0} \phi_t^{\otimes N} \frac{e^{-\frac{\|\phi_0\|^2}{2}}}{\sqrt{N!}} = E(\phi_t)$$

with ϕ_t solution to the Hartree equation:

$$i\partial_t \phi_t = -\Delta \phi_t + (v * |\phi_t|^2)\phi_t \,.$$

Quasifree States

The definition of Quasifree states is technical: omitted.
Key facts about quasifree states:

- States on the Fock space,
  i.e. positive linear forms ω^{qf} on the bounded linear operators on $\mathcal{F}\big(L^2(\mathbb{R}^d)\big) = \bigoplus_{N=0}^{\infty} L^2_s(\mathbb{R}^{dN})$, s. t. $\omega^{qf}(Id_{\mathcal{F}}) = 1$.
- If $L^2(\mathbb{R}^d)$ is replaced by $\mathbb{C}$,
  then $\mathcal{F}(\mathbb{C}) \cong Barg(\mathbb{C}) \subset L^2(\mathbb{C})$,
  and quasifree states are "gaussians" in $L^2(\mathbb{C})$.
- Characterized by their (truncated) moments of order 1 and 2.

(Order 1) $\phi \in L^2(\mathbb{R}^d)$,
(Order 2) $\gamma \in \mathfrak{S}^1\big(L^2(\mathbb{R}^d)\big)$, $\gamma^* = \gamma \geq 0$,
(Order 2) $\sigma \in L^2(\mathbb{R}^{2d}) \simeq \mathfrak{S}^2\big(L^2(\mathbb{R}^d)\big)$, $\sigma^T = \sigma$.

Equivalence of the three dynamics

Theorem (Bach, B., Chen, Fröhlich, Sigal, 2016)

Assume $v^2 \lesssim 1 - \Delta$, $v(x) = v(-x)$,
and, with $M = (1-\Delta)^{1/2}$, $\phi_t \in H^1(\mathbb{R}^d)$,
$\gamma_t \in M^{-1}\,\mathfrak{S}^1\big(L^2(\mathbb{R}^d)\big)\,M^{-1}$, $\sigma_t \in H^1(\mathbb{R}^{2d})$, $\gamma^* = \gamma \geq 0$, $\sigma^T = \sigma$.
The we have equivalence of the three dynamics:

1. *For all observable* $\mathbb{A}$ *at most quadratic in the fields,*

$$i\partial_t \omega_t^{qf}(\mathbb{A}) = \omega_t^{qf}\left([\mathbb{A}, \mathbb{H}]\right).$$

2. $(\phi_t, \gamma_t, \sigma_t)$ *satisfy the HFB equations (given in the next slide)*
3. *For all observable* $\mathbb{A}$

$$i\partial_t \omega_t^{qf}(\mathbb{A}) = \omega_t^{qf}\left([\mathbb{A}, \mathbb{H}^{(2)}_{HFB}(\omega_t^{qf})]\right),$$

with $\mathbb{H}^{(2)}_{HFB}(\omega_t^{qf}) = \cdots$.

Hartree-Fock-Bogoliubov Equations for Bosons

Hartree-Fock-Bogoliubov Equations for Bosons:

$$i\partial_t \phi_t = h(\gamma_t)\phi_t + k(\sigma_t)\bar{\phi}_t + (v * |\phi_t|^2)\phi_t \ ,$$
$$i\partial_t \gamma_t = [h(\gamma_t^{\phi_t}), \gamma_t]_- + [k(\sigma_t^{\phi_t}), \sigma_t]_- \ ,$$
$$i\partial_t \sigma_t = [h(\gamma_t^{\phi_t}), \sigma_t]_+ + [k(\sigma_t^{\phi_t}), \gamma_t]_+ + k(\sigma_t^{\phi_t}) \ ,$$

where $(v\sharp A)(x;y) := v(x-y)A(x;y)$, $A \in \mathcal{B}(L^2(\mathbb{R}^d))$ and

$$h(\gamma) := -\Delta + v * \gamma(x;x) + v\sharp\gamma \, , \qquad k(\sigma) := v\sharp\sigma \, ,$$
$$\gamma_t^{\phi_t} := \gamma_t + |\phi_t\rangle\langle\phi_t| \, , \qquad \sigma_t^{\phi_t} := \sigma_t + \phi_t \otimes \phi_t \, ,$$
$$[A,B]_- := AB^* - BA^* \, , \qquad [A,B]_+ := AB^T + BA^T \, .$$

Existence and Uniqueness

Theorem (Bach, B., Chen, Fröhlich, Sigal, 2016)

Assume $v^2 \lesssim 1 - \Delta$, $v(x) = v(-x)$,
and $\phi_0 \in H^1(\mathbb{R}^d)$, $\gamma_0 \in M^{-1}\,\mathfrak{S}^1\big(L^2(\mathbb{R}^d)\big)\,M^{-1}$, $\sigma_0 \in H^1(\mathbb{R}^{2d})$,
Then there is existence and uniqueness of maximal mild solutions $(\phi_t, \gamma_t, \sigma_t)$ *to the HFB equations.*
If moreover

$$\begin{pmatrix} \gamma_0 & \sigma_0 \\ \sigma_0^* & 1 + \bar{\gamma}_0 \end{pmatrix} \geq 0\,,$$

then the solution is global.

Preservation Laws

Theorem (Bach, B., Chen, Fröhlich, Sigal, 2016)

Assume $v^2 \lesssim 1 - \Delta$, $v(x) = v(-x)$,
and $\phi_t \in H^1(\mathbb{R}^d)$, $\gamma_t \in M^{-1}\,\mathfrak{S}^1\big(L^2(\mathbb{R}^d)\big)\,M^{-1}$, $\sigma_t \in H^1(\mathbb{R}^{2d})$.
Then

- *the energy* $\omega_t^{qf}(\mathbb{H})$ *is preserved by the HFB equations,*
- *the number of particles is preserved,*
- *for any obervable* $\mathbb{A}$ *at most quadratic in the field and such that* $[\mathbb{A}, \mathbb{H}] = 0$, $\omega_t^{qf}(\mathbb{A})$ *is preserved.*

Previous Works

1996 Griffin. *Conserving and gapless approximations for an inhomogeneous Bose gas at finite temperatures.*

1998 Dodd, Edwards, Clark, and Burnett. *Collective excitations of Bose-Einstein-condensed gases at finite temperatures.*

1998 Parkins and Walls. *The physics of trapped dilute-gas Bose–Einstein condensates.*

2010/2011 Grillakis, Machedon, and Margetis. *Second-order corrections to mean field evolution of weakly interacting bosons. I & II.*

2015/arxiv Grillakis and Machedon. *Pair excitations and the mean field approximation of interacting bosons, I & II.*

2015 Lewin, Nam, and Schlein. *Fluctuations around Hartree states in the mean-field regime.*

arxiv Nam and Napiórkowski. *Bogoliubov correction to the mean-field dynamics of interacting bosons.*

arxiv Lewin. *Mean-field limit of bose systems: rigorous results.*

arxiv Boccato, Cenatiempo and Schlein. *Quantum many-body fluctuations around nonlinear Schrödinger dynamics.*

Thank you for your attention.