Topological nature of the Fu-Kane-Mele invariants

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Outline

1. Time reversal symmetries and "Quaternionic" structures
2. The role of the (involutive) base space
3. In the search of a classifying object
4. FKMM vs. Fu-Kane-Mele

Topological Quantum Systems with odd TRS's

*Let $\mathbb{B}$ a topological space, (**"Brillouin zone"**). Assume that:*

- *$\mathbb{B}$ is compact, Hausdorff and path-connected;*
- *$\mathbb{B}$ admits a CW-complex structure.*

DEFINITION (*Topological Quantum System (TQS)*)

*Let $\mathcal{H}$ be a separable Hilbert space and $\mathcal{K}(\mathcal{H})$ the algebra of compact operators. A **TQS** is a self-adjoint map*

$$\mathbb{B} \ni k \longmapsto H(k) = H(k)^* \in \mathcal{K}(\mathcal{H})$$

***continuous** with respect to the norm-topology.*

- *The **spectrum** $\sigma(H(k)) = \{E_j(k) \mid j \in \mathcal{I} \subseteq \mathbb{Z}\} \subset \mathbb{R}$, is a sequence of eigenvalues ordered according to*

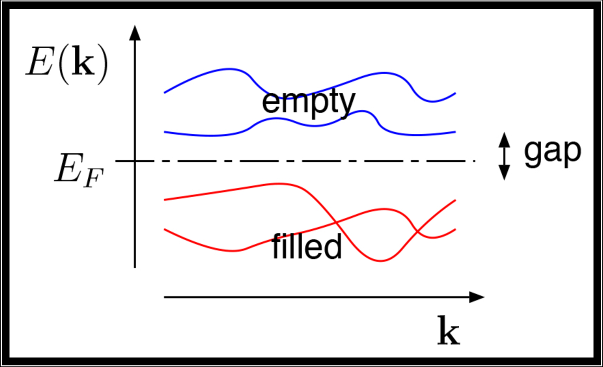
$$\ldots E_{-2}(k) \leqslant E_{-1}(k) < 0 \leqslant E_1(k) \leqslant E_2(k) \leqslant \ldots$$

- *The maps $k \mapsto E_j(k)$ are **continuous** (energy bands) ...*

Topological Quantum Systems with odd TRS's

... namely a band spectrum

$$H(k)\,\psi_j(k) \;=\; E_j(k)\,\psi_j(k)\,, \qquad\qquad k \in \mathbb{B}$$



☞ *Usually an energy **gap** separates the filled **valence** bands from the empty **conduction** bands. The **Fermi level** E_F characterizes the gap.*

Topological Quantum Systems with odd TRS's

A homeomorphism $\tau : \mathbb{B} \to \mathbb{B}$ *is called* ***involution*** *if* $\tau^2 = \mathrm{Id}_{\mathbb{B}}$. *The pair* $(\mathbb{B}, \tau)$ *is called an* ***involutive space***. *Each space* $\mathbb{B}$ *admits (at least) the* ***trivial involution*** $\tau_{\mathrm{triv}} := \mathrm{Id}_{\mathbb{B}}$.

DEFINITION (*TQS with time-reversal symmetry***)**

Let $(\mathbb{B}, \tau)$ *be an involutive space,* $\mathcal{H}$ *a separable Hilbert space endowed with a* ***complex conjugation*** C. *A TQS* $\mathbb{B} \ni k \mapsto H(k)$ *has a* ***time-reversal symmetry*** *(TRS) of parity* $\eta \in \{\pm 1\}$ *if there is a continuous unitary-valued map* $k \mapsto U(k)$ *such that*

$$U(k)\; H(k)\; U(k)^* = C\; H(\tau(k))\; C\,, \qquad C\; U(\tau(k))\; C = \eta U(k)^*\;.$$

A TQS with an ***odd*** *TRS (i.e.* $\eta = -1$*) is called of class* **AII**.

The Serre-Swan construction

- *An **isolated family** of energy bands is any (finite) collection* $\{E_{j_1}(\cdot), \ldots, E_{j_m}(\cdot)\}$ *of energy bands such that*

$$\min_{k \in \mathbb{B}} \operatorname{dist}\left(\bigcup_{s=1}^{m} \{E_{j_s}(k)\} \, , \bigcup_{j \in \mathcal{I} \setminus \{j_1, \ldots, j_m\}} \{E_j(k)\} \right) = C_g > 0 \, .$$

*This is usually called **gap condition**.*

- *An isolated family is described by the **Fermi projection***

$$P_F(k) := \sum_{s=1}^{m} |\psi_{j_s}(k)\rangle\langle\psi_{j_s}(k)| \, .$$

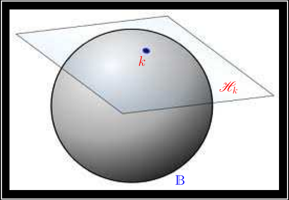
*This is a **continuous** projection-valued map*

$$\mathbb{B} \ni k \longmapsto P_F(k) \in \mathcal{K}(\mathcal{H}) \, .$$

The Serre-Swan construction

- *For each* $k \in \mathbb{B}$
$$\mathcal{H}_k := \mathrm{Ran}\, P_F(k) \subset \mathcal{H}$$
is a subspace of $\mathcal{H}$ *of* ***fixed*** *dimension* m.

- *The collection*
$$\mathcal{E}_F := \bigsqcup_{k \in \mathbb{B}} \mathcal{H}_k$$
is a topological space (said ***total space****) and the* ***map***
$$\pi : \mathcal{E}_F \longrightarrow \mathbb{B}$$
defined by $\pi(k, \mathrm{v}) = k$ *is continuous (and open).*



This is a ***complex*** *vector bundle (of rank* m*) called* ***Bloch-bundle****.*

The Serre-Swan construction

☞ *An **odd** TRS induces a **"Quaternionic"** structure on the Bloch-bundle.*

DEFINITION (*Atiyah, 1966 - Dupont, 1969***)**

Let $(\mathbb{B}, \tau)$ *be an involutive space and* $\mathscr{E} \to \mathbb{B}$ *a* ***complex*** *vector bundle. Let* $\Theta : \mathscr{E} \to \mathscr{E}$ *an* ***homeomorphism*** *such that*

$$\Theta \ : \ \mathscr{E}|_k \longrightarrow \mathscr{E}|_{\tau(k)} \qquad \text{is } \textbf{anti}\text{-linear} .$$

$[\mathcal{R}]$ - *The pair* $(\mathscr{E}, \Theta)$ *is a* ***"Real"****-bundle over* $(\mathbb{B}, \tau)$ *if*

$$\Theta^2 \ : \ \mathscr{E}|_k \xrightarrow{+1} \mathscr{E}|_k \qquad \forall\ k \in \mathbb{B} ;$$

$[\mathcal{Q}]$ - *The pair* $(\mathscr{E}, \Theta)$ *is a* ***"Quaternionic"****-bundle over* $(\mathbb{B}, \tau)$ *if*

$$\Theta^2 \ : \ \mathscr{E}|_k \xrightarrow{-1} \mathscr{E}|_k \qquad \forall\ k \in \mathbb{B} .$$

The classification problem

DEFINITION (*Topological phases*)

Let $\mathbb{B} \ni k \longmapsto H(k)$ *be an* ***odd TR-symmetric*** *TQS with an isolated family of* m *energy bands and associated* ***"Quaternionic"*** *Bloch bundle* $\mathscr{E}_F \longrightarrow \mathbb{B}$. *The* ***topological phase*** *of the system is specified by*

$$[(\mathscr{E}_F, \Theta)] \in \mathrm{Vec}^m_{\mathcal{Q}}(\mathbb{B}, \tau) \, .$$

$\Downarrow\Downarrow\Downarrow$

Main Question:

How to classify $\mathrm{Vec}^m_{\mathcal{Q}}(\mathbb{B}, \tau)$ *at least for* ***low-dimensional*** $\mathbb{B}$*?*

The classification problem

Known results for $\dim(\mathbb{B}) \leqslant 3$

- $\mathrm{Vec}^m_{\mathbb{C}}(\mathbb{B}) \overset{c_1}{\simeq} H^2(\mathbb{B}, \mathbb{Z})$ *(Peterson, 1959)*
- $\mathrm{Vec}^m_{\mathcal{R}}(\mathbb{B}, \tau) \overset{c_1^{\mathcal{R}}}{\simeq} H^2_{\mathbb{Z}_2}(\mathbb{B}, \mathbb{Z}(1))$ *(Kahn, 1987 - D. & Gomi, 2014)*

CAZ	TRS	Category	VB
A	0	*complex*	$\mathrm{Vec}^m_{\mathbb{C}}(\mathbb{B})$
AI	$+$	*"Real"*	$\mathrm{Vec}^m_{\mathcal{R}}(\mathbb{B}, \tau)$
AII	$-$	*"Quaternionic"*	$\mathrm{Vec}^m_{\mathcal{Q}}(\mathbb{B}, \tau)$

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Electrons in a periodic environment

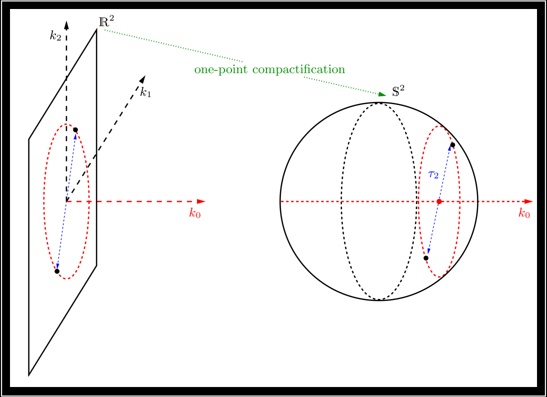
- **Periodic** quantum systems (e.g. absence of **disorder**):
  - $\mathbb{R}^d$-translations $\Rightarrow$ **free** (Dirac) fermions;
  - $\mathbb{Z}^d$-translations $\Rightarrow$ **crystal** (Bloch) fermions.
- The Bloch-Floquet (or Fourier) theory exploits the invariance under translations of a periodic structure to describe the state of the system in terms of the **quasi-momentum** k on the **Brillouin zone** $\mathbb{B}$.
- Complex conjugation (TRS) endows $\mathbb{B}$ with an involution τ.
- Examples are:
  - Gapped electronic systems,
  - BdG superconductors,
  - **Photonic crystals** (M. Lein talk).

Continuous case $\mathbb{B} \equiv \mathbb{S}^{1,d}$

$$\mathbb{S}^d \xrightarrow{\theta_{1,d}} \mathbb{S}^d$$

$$\mathbb{S}^{1,d} := (\mathbb{S}^d, \theta_{1,d})$$

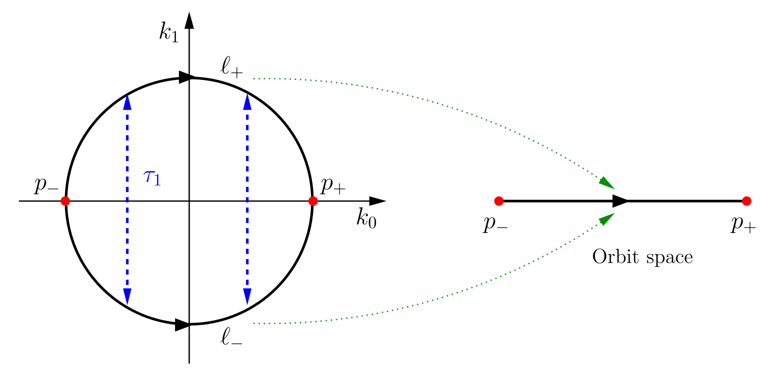
$$(+k_0, +k_1, \ldots, +k_d) \overset{\theta_{1,d}}{\longmapsto} (+k_0, -k_1, \ldots, -k_d)$$



Periodic case $\mathbb{B} \equiv \mathbb{T}^{0,d,0}$

$$\mathbb{S}^{1,1} \times \ldots \times \mathbb{S}^{1,1} \xrightarrow{\tau_d := \theta_{1,1} \times \ldots \times \theta_{1,1}} \mathbb{S}^{1,1} \times \ldots \times \mathbb{S}^{1,1}$$

$$\mathbb{T}^{0,d,0} := \underbrace{\mathbb{S}^{1,1} \times \ldots \times \mathbb{S}^{1,1}}_{d\text{ - times}} = (\mathbb{T}^d, \tau_d)$$



Topological states for Bloch electrons

	$d = 1$	$d = 2$	$d = 3$	$d = 4$	
$\mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{S}^{1,d})$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	*Free*
$\mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{T}^{0,d,0})$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^{10} \oplus \mathbb{Z}$	*Periodic*

- *The first proof (for the case* $d = 1, 2$*) is due to* ***Fu****,* ***Kane*** *and* ***Mele*** ***(2005 - 2007)****. They introduced the notion of* ***Fu-Kane-Mele indices*** *(values of a Pfaffian on the fixed points) and the distinction between* ***strong*** *and* ***weak*** *invariants.*
- *Computed by* ***Kitaev (2009)*** *for all* d *by* ***K-theory*** *(stable range).*
- *"Handmade"* ***frame construction*** *for the case* $\mathbb{T}^{0,2,0}$ *by* ***Graf*** *and* ***Porta*** ***(2013)*** *and for the case* $\mathbb{T}^{0,3,0}$ *by* ***Fiorenza****,* ***Monaco*** *and* ***Panati (2016)****.*
- ***Kennedy*** *and* ***Zirnbauer (2015)*** *by the calculation of the* ***equivariant homotopy*** *(very general but* ***hard*** *to compute).*
- ***D.*** *and* ***Gomi (2015)*** *by the introduction of the* ***FKMM-invariant*** *(a characteristic class) and the computation of the* ***equivariant cohomology*** *(very general and* ***not so hard*** *to compute).*

Why more general involutive spaces?

- $\mathbb{B}$ can be interpreted as the space of **control parameters** for a quantum system **adiabatically perturbed**. In this sense $(\mathbb{B}, \tau)$ can be very general. In particular the **fixed-point set** $\mathbb{B}^\tau$ could be empty (free action) or a sub-manifold of whatever co-dimension (and not necessary a discrete set of points).

- Many of the previous approaches just fail when $\mathbb{B}^\tau$ is not a discrete set: e. g. **which is the meaning of the Fu-Kane-Mele indices when $\mathbb{B}^\tau$ is not a discrete set?**

- Recently **Gat** and **Robbins (arXiv:1511.08994)** considered the cases $\mathbb{B} = \mathbb{S}^{0,3}$ (**rigid rotor**) and $\mathbb{B} = \mathbb{T}^{1,1,0}$ (**phase space of slow dynamic of a 1D periodic particle**). In the first case $\mathbb{B}^\tau = \emptyset$ and in the second $\mathbb{B}^\tau = \mathbb{S}^1 \sqcup \mathbb{S}^1$. The classification is obtained by a "handmade" frame construction:

$$\mathrm{Vec}^m_{\mathcal{Q}}(\mathbb{S}^{0,3}) \simeq \begin{cases} 2\mathbb{Z}+1 & m \ \text{odd} \\ 2\mathbb{Z} & m \ \text{even} \end{cases}, \qquad \mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{T}^{1,1,0}) \simeq 2\mathbb{Z}\,.$$

- **11** pages for **2** cases ... but there are much more !!

More general involutive spheres

$\mathbb{S}^{p,q} := (\mathbb{S}^{p+q-1}, \theta_{p,q})$ *with* $\theta_{p,q}$ *defined by*

$$(k_0, k_1, \ldots, k_{p-1}, k_p, \ldots, k_{p+q-1}) \overset{\theta_{p,q}}{\mapsto} (k_0, k_1, \ldots, k_{p-1}, -k_p, \ldots, -k_{p+q-1})$$

$p+q \leqslant 4$	$q=0$	$q=1$	$q=2$	$q=3$	$q=4$
$\mathrm{Vec}_{\mathcal{Q}}^{2m+1}(\mathbb{S}^{0,q})$	$\emptyset$	?	?	$2\mathbb{Z}+1$	?
$\mathrm{Vec}_{\mathcal{Q}}^{2m}(\mathbb{S}^{0,q})$	$\emptyset$	?	?	$2\mathbb{Z}$	?
$\mathrm{Vec}_{\mathcal{Q}}^{2m}(\mathbb{S}^{1,q})$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	...
$\mathrm{Vec}_{\mathcal{Q}}^{2m}(\mathbb{S}^{2,q})$	0	?	?	...	
$\mathrm{Vec}_{\mathcal{Q}}^{2m}(\mathbb{S}^{3,q})$	0	?	...		
$\mathrm{Vec}_{\mathcal{Q}}^{2m}(\mathbb{S}^{4,q})$	0	...			

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$p+q \leqslant 4$	$q=0$	$q=1$	$q=2$	$q=3$	$q=4$
$\mathrm{Vec}^{2m+1}_{\mathcal{Q}}(\mathbb{S}^{0,q})$	$\emptyset$	0	0	$2\mathbb{Z}+1$	$\emptyset$
$\mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{S}^{0,q})$	$\emptyset$	0	0	$2\mathbb{Z}$	0
$\mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{S}^{1,q})$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\ldots$
$\mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{S}^{2,q})$	0	$2\mathbb{Z}$	0	$\ldots$	
$\mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{S}^{3,q})$	0	0	$\ldots$		
$\mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{S}^{4,q})$	0	$\ldots$			

More general involutive tori (fixed-point case)

$$\mathbb{T}^{a,b,c} := \underbrace{\mathbb{S}^{2,0} \times \ldots \times \mathbb{S}^{2,0}}_{a-\textit{times}} \times \underbrace{\mathbb{S}^{1,1} \times \ldots \times \mathbb{S}^{1,1}}_{b-\textit{times}} \times \underbrace{\mathbb{S}^{0,2} \times \ldots \times \mathbb{S}^{0,2}}_{c-\textit{times}}$$

$a + b \leqslant 3,\ c = 0$	$a = 0$	$a = 1$	$a = 2$	$a = 3$
$\mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{T}^{a,0,0})$	$\emptyset$	0	0	0
$\mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{T}^{a,1,0})$	0	$2\mathbb{Z}$	?	...
$\mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{T}^{a,2,0})$	$\mathbb{Z}_2$	?	...	
$\mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{T}^{a,3,0})$	$\mathbb{Z}_2^4$	...		

More general involutive tori (fixed-point case)

$$\mathbb{T}^{a,b,c} := \underbrace{\mathbb{S}^{2,0} \times \ldots \times \mathbb{S}^{2,0}}_{a-times} \times \underbrace{\mathbb{S}^{1,1} \times \ldots \times \mathbb{S}^{1,1}}_{b-times} \times \underbrace{\mathbb{S}^{0,2} \times \ldots \times \mathbb{S}^{0,2}}_{c-times}$$

$a+b \leqslant 3,\ c=0$	$a=0$	$a=1$	$a=2$	$a=3$
$\mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{T}^{a,0,0})$	$\emptyset$	0	0	0
$\mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{T}^{a,1,0})$	0	$2\mathbb{Z}$	$(2\mathbb{Z})^2$	$\ldots$
$\mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{T}^{a,2,0})$	$\mathbb{Z}_2$	$\mathbb{Z}_2 \oplus (2\mathbb{Z})^2$	$\ldots$	
$\mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{T}^{a,3,0})$	$\mathbb{Z}_2^4$	$\ldots$		

More general involutive tori (free-involution case)

PROPOSITION (D. - Gomi, 2016)

$$\mathbb{T}^{a,b,c} \simeq \mathbb{T}^{a+c-1,b,1} \qquad \forall\, c \geqslant 2$$

$a+b \leqslant 2,\ c=1$	$a=0$	$a=1$	$a=2$
$\mathrm{Vec}^m_{\mathcal{Q}}(\mathbb{T}^{a,0,1})$	0	?	?
$\mathrm{Vec}^m_{\mathcal{Q}}(\mathbb{T}^{a,1,1})$	?	?	...
$\mathrm{Vec}^m_{\mathcal{Q}}(\mathbb{T}^{a,2,1})$	?	...	

For all $m \in \mathbb{N}$ *odd or even!*

More general involutive tori (free-involution case)

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$\mathrm{Vec}^m_{\mathcal{Q}}(\mathbb{T}^{a,0,1})$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$
$\mathrm{Vec}^m_{\mathcal{Q}}(\mathbb{T}^{a,1,1})$	$2\mathbb{Z}$	$\mathbb{Z}_2 \oplus (2\mathbb{Z})^2$	$\ldots$
$\mathrm{Vec}^m_{\mathcal{Q}}(\mathbb{T}^{a,2,1})$	$(2\mathbb{Z})^2$	$\ldots$	

For all $m \in \mathbb{N}$ *odd or even!*

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Relative equivariant cohomology

In ***[D. - Gomi, 2016]*** *we classified*

$$\mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{T}^{0,d,0}) \qquad \textit{and} \qquad \mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{S}^{1,d-1})\,, \quad d \leqslant 4$$

by a ***characteristic class*** *with values in* $H^2_{\mathbb{Z}_2}(\mathbb{B}|\mathbb{B}^\tau, \mathbb{Z}(1))$*: the* ***FKMM-invariant****.*

$$\begin{array}{ccccccc} H^1_{\mathbb{Z}_2}(\mathbb{B}^\tau, \mathbb{Z}(1)) & \xrightarrow{\delta_1} & H^2(\mathbb{B}|\mathbb{B}^\tau, \mathbb{Z}(1)) & \xrightarrow{\delta_2} & H^2_{\mathbb{Z}_2}(\mathbb{B}, \mathbb{Z}(1)) & \xrightarrow{r} & H^2_{\mathbb{Z}_2}(\mathbb{B}^\tau, \mathbb{Z}(1)) \\ \wr\vert & & & & \wr\vert & & \wr\vert \\ [\mathbb{B}^\tau, \mathbb{S}^{1,1}]_{\mathbb{Z}_2} & & & & \mathrm{Pic}_{\mathcal{R}}(\mathbb{B}, \tau) & & \mathrm{Pic}_{\mathbb{R}}(\mathbb{B}^\tau) \end{array}$$

☞ *Our previous results only apply to the case*

$$\mathbb{B}^\tau \;=\; \{\textit{finite collection of points}\}.$$

To consider more general involutive spaces we need more generality !

The (generalized) FKMM-invariant

THEOREM (D. - Gomi, 2016)

Given $(\mathbb{B},\tau)$ *let*

$$\mathrm{Pic}_{\mathcal{R}}\big(\mathbb{B}|\mathbb{B}^{\tau},\tau\big) \;:=\; \{[(\mathscr{L},s)] \mid \mathscr{L}\in \mathrm{Pic}_{\mathcal{R}}(\mathbb{B},\tau)\;,\quad s:\mathscr{L}|_{\mathbb{B}^{\tau}}\to \mathbb{U}(1)\}\;.$$

The choice of s *is* ***canonical*** *and the group structure is given by the* ***tensor product****. Then*

$$\mathrm{Pic}_{\mathcal{R}}\big(\mathbb{B}|\mathbb{B}^{\tau},\tau\big) \;\overset{\tilde{\kappa}}{\simeq}\; H^2\big(\mathbb{B}|\mathbb{B}^{\tau},\mathbb{Z}(1)\big)\;.$$

There is a group homomorphism

$$\kappa\;:\;\mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{B},\tau)\;\longrightarrow H^2\big(\mathbb{B}|\mathbb{B}^{\tau},\mathbb{Z}(1)\big)$$

called the ***FKMM-invariant****.*

- *If* $(\mathscr{E},\Theta)\in \mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{B},\tau)$ *then* $(\det\mathscr{E},\det\Theta)\in \mathrm{Pic}_{\mathcal{R}}(\mathbb{B},\tau)$*;*
- *It exists a* ***canonical*** $s_{\mathscr{E}}:\mathbb{B}^{\tau}\to \det\mathscr{E}|_{\mathbb{B}^{\tau}}$
- $(\det\mathscr{E},s_{\mathscr{E}})\in \mathrm{Pic}_{\mathcal{R}}\big(\mathbb{B}|\mathbb{B}^{\tau},\tau\big)$

$$\kappa(\mathscr{E},\Theta)\;:=\;\tilde{\kappa}(\det\mathscr{E},s_{\mathscr{E}})\;.$$

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$$\kappa \ : \ \mathrm{Vec}^{2m}_{\mathcal{Q}}(\mathbb{B},\tau) \longrightarrow H^2\big(\mathbb{B}|\mathbb{B}^\tau,\mathbb{Z}(1)\big)$$

- ***Isomorphic*** *$\mathcal{Q}$-bundles have the same FKMM-invariant;*
- *If $(\mathscr{E},\Theta)$ is $\mathcal{Q}$-**trivial** then $\kappa(\mathscr{E},\Theta)=0$;*
- *κ is **natural** under the pullback induced by equivariant maps;*
- $\kappa(\mathscr{E}_1\oplus\mathscr{E}_2,\Theta_1\oplus\Theta_2) \ = \ \kappa(\mathscr{E}_1,\Theta_1) \ + \ \kappa(\mathscr{E}_2,\Theta_2)$;
- *κ is the image of a **universal class** $\mathfrak{h}_{univ}$;*
- *When $\mathbb{B}^\tau \ = \ \{$finite collection of points$\}$*

$$\kappa(\mathscr{E},\Theta) \ \simeq \ \textit{Fu-Kane-Mele invariants} \ ;$$

- *When $\mathbb{B}^\tau \ = \ \emptyset$*

$$\kappa(\mathscr{E},\Theta) \ \simeq \ c_1^{\mathcal{R}}(\det\mathscr{E},\det\Theta) \ ;$$

- *If $\dim(\mathbb{B})\leqslant 3$ the map κ is **injective**;*
- *Over (low dimensional) $\mathbb{S}^{p,q}$ and $\mathbb{T}^{a,b,c}$ the map κ is **bijective**;*
- *κ is **not bijective** in general, neither in low dimension ... damn!!;*
- *When $\mathbb{B}^\tau \ = \ \emptyset$ and $\mathrm{Pic}_{\mathcal{Q}}(\mathbb{B},\tau) \ = \ \emptyset$ then $\mathrm{Pic}_{\mathcal{Q}}(\mathbb{B},\tau)$ is a **torsor** over $\mathrm{Pic}_{\mathcal{R}}(\mathbb{B},\tau)$. Hence*

$$\mathrm{Pic}_{\mathcal{Q}}(\mathbb{B},\tau) \ \simeq \ \mathrm{Pic}_{\mathcal{R}}(\mathbb{B},\tau) \ \simeq \ H^2(\mathbb{B}|\mathbb{B}^\tau,\mathbb{Z}(1)) \ .$$

Thank you for your attention