Zeta regularization and the Casimir effect: a functional analytic framework.

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1. General background and basic ideas.

Zeta Regularization (ZR): to give meaning via analytic continuation to ill-defined expressions appearing in mathematics and physics.

(Minakshisundaram and Pleijel, 1945; Seeley, 1967; Ray and Singer, 1971)

- A textbook example ($\zeta =$ Riemann zeta function):

$$\sum_{n=1}^{+\infty} n \text{ ``=''} \left[\sum_{n=1}^{+\infty} \frac{1}{n^s}\right]_{s=-1} \text{ ``=''} \Big[\zeta(s)\Big]_{s=-1} = -\frac{1}{12} \, .$$

- Often used to treat the divergences of quantum field theory (QFT), especially, in connection with vacuum effects.

  (Dowker and Critchley, 1975; Hawking, 1977; Wald, 1979; Bytsenko, Cognola, Elizalde, Kirsten, Moretti, Vanzo, Zerbini, 1985-today)

Casimir effect (CE): physical phenomena related to the vacuum state of a quantum field interacting with classical boundaries/potentials.

$\hookrightarrow$ *Study CE using ZR*.
Case study: **Hermitian scalar field**.

- **Rigorous viewpoint** on many divergent expressions in QFT : the field is an operator valued distribution ⇒ pointwise evaluation of field polynomials (and of corresponding VEVs) is ill-defined.
- **Standard zeta approach**:
  - Euclidean formulation in terms of (*formal*) functional integrals;
  - *typical assumption*: elliptic operators with pure point spectrum $\hookrightarrow$ ZR implemented via eigenfunction expansion techniques.
- **Novel results**:
  - fully rigorous analytic setting for ZR in the framework of Wightman quantization
  - point and continuous spectrum handled without differences;
  - analysis of some exactly solvable cases.

References

- D.F., L. Pizzocchero, **Prog.Teor.Phys. 126(3), 419–434** (2011);
- D.F., L. Pizzocchero, **arXiv:1505.00711**, **arXiv:1505.01044**, **Int.J.Mod.Phys.A30(35),1550213**, **Int.J.Mod.Phys.A31,1650003** (2015);
- D.F., PhD thesis (2016).

2. Functional analytic framework.

- **Basic elements**:
  - $(\mathcal{H}, \langle\ |\ \rangle) \equiv \mathcal{H}$ = abstract separable Hilbert space;
  - $\mathcal{A} : \mathrm{Dom}(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$, strictly positive, essentially self-adjoint.
- Spectral theorem $\to \mathcal{A}^{-s}$ $(s \in \mathbb{C})$;
  $\to e^{-\mathbf{t}\mathcal{A}}$ (*heat*), $e^{-\mathbf{t}\sqrt{\mathcal{A}}}$ (*cylinder*, [Fulling]) $(\mathbf{t} \in \mathbb{C})$.
- Consider the **inner products** $\langle g|f\rangle_r := \langle \mathcal{A}^{r/2} g | \mathcal{A}^{r/2} f\rangle$ $(r \in \mathbb{R})$:
  - $\mathcal{H}^r$:= completion of $\mathrm{Dom}(\mathcal{A}^{r/2})$ w.r.t. $\langle\ |\ \rangle_r$;
  - $\mathcal{H}^{+\infty} := \bigcap_{r\in\mathbb{R}} \mathcal{H}^r$ with *Fréchet topology*;
  - $\mathcal{H}^{-\infty} := \bigcup_{r\in\mathbb{R}} \mathcal{H}^r$ with *inductive limit topology*.

  $\Rightarrow$ $\mathcal{H}^0 = \mathcal{H}$ and $\mathcal{H}^r \overset{dense}{\hookrightarrow} \mathcal{H}^u$ if $r \geqslant u$ $(r, u \in \mathbb{R} \cup \{\pm\infty\})$.
- $\exists!\ \langle\ |\ \rangle : \bigcup_{r\in\mathbb{R}} \mathcal{H}^{-r} \times \mathcal{H}^r \to \mathbb{C}$ **extension of the inner product** on $\mathcal{H}$, s.t. $\langle\ |\ \rangle \restriction \mathcal{H}^{-r} \times \mathcal{H}^r$ $(r \in \mathbb{R})$ is a *continuous, sesquilinear Hermitian form*.

  $\Rightarrow$ $\mathcal{H}^{-r} \overset{isom}{\simeq} (\mathcal{H}^r)'$ = topol. dual of $\mathcal{H}^r$ $(r \in \mathbb{R} \cup \{+\infty\})$.

- $\exists!\, \mathcal{A}^{-s}, e^{-\mathbf{t}\mathcal{A}}, e^{-\mathbf{t}\sqrt{\mathcal{A}}} : \mathcal{H}^{-\infty} \to \mathcal{H}^{-\infty}$ $(s, \mathbf{t} \in \mathbb{C}$ with $\Re \mathbf{t} \geqslant 0)$
  continuous extensions of $\mathcal{A}^{-s}, e^{-\mathbf{t}\mathcal{A}}, e^{-\mathbf{t}\sqrt{\mathcal{A}}}$ on $\mathcal{H}$
  $\Rightarrow$ *restrictions to finite order spaces* fulfill norm bounds and depend analytically on the parameters.

- **Typical application**:
  - $\mathcal{H} = L^2(\Omega)$, with $\Omega \subset \mathbb{R}^d$ *any* open subset (or Riemannian manifold);
  - $\mathcal{A} := (-\triangle + V)\restriction \mathcal{D}_{\mathcal{A}}$, with $\triangle =$ Laplacian on $\mathbb{R}^d$, $V \in C^\infty(\Omega)$, $\mathcal{D}_{\mathcal{A}} \subset L^2(\Omega)$ a *suitable* domain (keeping into account b.c.).

  In this case: $\mathcal{H}^r \hookrightarrow H^r_{loc}(\Omega) \hookrightarrow C^j(\Omega)$ $\;(j \in \mathbb{N},\, r > j + d/2)$.

- $\exists!\, \delta_{\mathbf{x}} \in \mathcal{H}^{-\infty}$ $(\mathbf{x} \in \Omega)$ s.t. $\langle \delta_{\mathbf{x}} | f \rangle = f(\mathbf{x})$ $(f \in \mathcal{H}^r, r > d/2)$: **Dirac delta**.

- Let $\mathcal{B} : \mathrm{Dom}(\mathcal{B}) \subset \mathcal{H}^{-\infty} \to \mathcal{H}^{-\infty}$ and *assume* $\exists\, \theta > 0,\, j \in \mathbb{N}$ s.t., $\forall\, j_1 + j_2 \leqslant j$, $\mathcal{B} : \mathcal{H}^{-(j_2 + d/2 + \theta)} \to \mathcal{H}^{j_1 + d/2 + \theta}$ is continuous
  $\hookrightarrow$ the **integral kernel** of $\mathcal{B}$ is $\mathcal{B}(\mathbf{x}, \mathbf{y}) := \langle \delta_{\mathbf{x}} | \mathcal{B}\, \delta_{\mathbf{y}} \rangle$ $(\mathbf{x}, \mathbf{y} \in \Omega)$.
  $\Rightarrow (\mathcal{B}f)(\mathbf{x}) = \int_\Omega d\mathbf{y}\, \mathcal{B}(\mathbf{x}, \mathbf{y}) f(\mathbf{y})$ $(\forall f \in L^2(\Omega))$ and $\mathcal{B}(\;,\;) \in C^j(\Omega \times \Omega)$.

- The **Dirichlet kernel** is $\mathcal{A}^{-s}(\mathbf{x},\mathbf{y})$. For $\Re s > (j+d)/2$ one has $\mathcal{A}^{-s}(\ ,\ ) \in C^j(\Omega\times\Omega)$ and $s \mapsto \partial^\alpha_{\mathbf{x}}\partial^\beta_{\mathbf{y}}\mathcal{A}^{-s}(\mathbf{x},\mathbf{y})$ is analytic $(\mathbf{x},\mathbf{y}\in\Omega)$;
- The **heat** and **cylinder kernels** are $e^{-\mathbf{t}\mathcal{A}}(\mathbf{x},\mathbf{y})$, $e^{-\mathbf{t}\sqrt{\mathcal{A}}}(\mathbf{x},\mathbf{y})$ $(\Re\mathbf{t} > 0)$.
- Derive the **Mellin relations** (for $\Re s > d/2$)

$$\mathcal{A}^{-s}(\mathbf{x},\mathbf{y}) = \frac{1}{\Gamma(s)}\int_0^{+\infty} d\mathbf{t}\ \mathbf{t}^{s-1}\, e^{-\mathbf{t}\mathcal{A}}(\mathbf{x},\mathbf{y}) = \frac{1}{\Gamma(2s)}\int_0^{+\infty} d\mathbf{t}\ \mathbf{t}^{2s-1}\, e^{-\mathbf{t}\sqrt{\mathcal{A}}}(\mathbf{x},\mathbf{y})$$

(**holding also for $\mathbf{y}=\mathbf{x}\in\Omega$**; similar relations for $\partial^\alpha_{\mathbf{x}}\partial^\beta_{\mathbf{y}}\mathcal{A}^{-s}(\mathbf{x},\mathbf{y})$).

- Construct the **analytic continuation** of $\mathcal{A}^{-s}(\mathbf{x},\mathbf{y})$:

1) Standard method using the *heat kernel asymptotics*: if $\exists\, N\in\mathbb{N}$, $a_n : \Omega\times\Omega \to \mathbb{R}$ $(n=0,\dots,N)$, $r_N(\mathbf{t};\mathbf{x},\mathbf{y}) = O(\mathbf{t}^{N+1})$ $(\mathbf{t}\to 0^+)$ s.t. $e^{-\mathbf{t}\mathcal{A}}(\mathbf{x},\mathbf{y}) = \frac{1}{\mathbf{t}^{d/2}}\left(\sum_{n=0}^N a_n(\mathbf{x},\mathbf{y})\,\mathbf{t}^n + r_N(\mathbf{t};\mathbf{x},\mathbf{y})\right)$, then for $\Re s > d/2-(N+1)$

$$\mathcal{A}^{-s}(\mathbf{x},\mathbf{y}) =$$

$$\frac{1}{\Gamma(s)}\left(\sum_{n=0}^N \frac{a_n(\mathbf{x},\mathbf{y})}{s+n-\frac{d}{2}} + \int_0^1 d\mathbf{t}\ \mathbf{t}^{s-\frac{d}{2}-1} r_N(\mathbf{t};\mathbf{x},\mathbf{y}) + \int_1^{+\infty} d\mathbf{t}\ \mathbf{t}^{s-1} e^{-\mathbf{t}\mathcal{A}}(\mathbf{x},\mathbf{y})\right).$$

- The **Dirichlet kernel** is $\mathcal{A}^{-s}(\mathbf{x},\mathbf{y})$. For $\Re s > (j+d)/2$ one has $\mathcal{A}^{-s}(\ ,\ ) \in C^j(\Omega\times\Omega)$ and $s \mapsto \partial^\alpha_{\mathbf{x}} \partial^\beta_{\mathbf{y}} \mathcal{A}^{-s}(\mathbf{x},\mathbf{y})$ is analytic $(\mathbf{x},\mathbf{y}\in\Omega)$;
- The **heat** and **cylinder kernels** are $e^{-\mathbf{t}\mathcal{A}}(\mathbf{x},\mathbf{y})$, $e^{-\mathbf{t}\sqrt{\mathcal{A}}}(\mathbf{x},\mathbf{y})$ $(\Re\mathbf{t} > 0)$.
- Derive the **Mellin relations** (for $\Re s > d/2$)

$$\mathcal{A}^{-s}(\mathbf{x},\mathbf{y}) = \frac{1}{\Gamma(s)} \int_0^{+\infty} d\mathbf{t}\ \mathbf{t}^{s-1}\, e^{-\mathbf{t}\mathcal{A}}(\mathbf{x},\mathbf{y}) = \frac{1}{\Gamma(2s)} \int_0^{+\infty} d\mathbf{t}\ \mathbf{t}^{2s-1}\, e^{-\mathbf{t}\sqrt{\mathcal{A}}}(\mathbf{x},\mathbf{y})$$

(**holding also for** $\mathbf{y} = \mathbf{x} \in \Omega$; similar relations for $\partial^\alpha_{\mathbf{x}} \partial^\beta_{\mathbf{y}} \mathcal{A}^{-s}(\mathbf{x},\mathbf{y})$).

- Construct the **analytic continuation** of $\mathcal{A}^{-s}(\mathbf{x},\mathbf{y})$:

2) *Integration by parts*: if $\exists\, N \in \mathbb{N}$, $H : [0,+\infty)\times\Omega\times\Omega \to \mathbb{R}$ s.t. $H(\ ;\mathbf{x},\mathbf{y}) \in C^N([0,+\infty))$ $(\mathbf{x},\mathbf{y}\in\Omega)$, $e^{-\mathbf{t}\mathcal{A}}(\mathbf{x},\mathbf{y}) = \frac{1}{\mathbf{t}^{d/2}} H(\mathbf{t};\mathbf{x},\mathbf{y})$, then for $\Re s > d/2 - N$

$$\mathcal{A}^{-s}(\mathbf{x},\mathbf{y}) = \frac{(-1)^N}{(s-\frac{d}{2})\ldots(s-\frac{d}{2}+N-1)\,\Gamma(s)} \int_0^{+\infty} d\mathbf{t}\ \mathbf{t}^{s-\frac{d}{2}+N-1}\, \partial^N_{\mathbf{t}} H(\mathbf{t};\mathbf{x},\mathbf{y})\ .$$

- The **Dirichlet kernel** is $\mathcal{A}^{-s}(\mathbf{x},\mathbf{y})$. For $\Re s > (j+d)/2$ one has $\mathcal{A}^{-s}(\ ,\ ) \in C^j(\Omega\times\Omega)$ and $s \mapsto \partial^\alpha_{\mathbf{x}}\partial^\beta_{\mathbf{y}}\mathcal{A}^{-s}(\mathbf{x},\mathbf{y})$ is analytic $(\mathbf{x},\mathbf{y}\in\Omega)$;
- The **heat** and **cylinder kernels** are $e^{-\mathbf{t}\mathcal{A}}(\mathbf{x},\mathbf{y})$, $e^{-\mathbf{t}\sqrt{\mathcal{A}}}(\mathbf{x},\mathbf{y})$ $(\Re\mathbf{t} > 0)$.
- Derive the **Mellin relations** (for $\Re s > d/2$)

$$\mathcal{A}^{-s}(\mathbf{x},\mathbf{y}) = \frac{1}{\Gamma(s)}\int_0^{+\infty} d\mathbf{t}\ \mathbf{t}^{s-1}\, e^{-\mathbf{t}\mathcal{A}}(\mathbf{x},\mathbf{y}) = \frac{1}{\Gamma(2s)}\int_0^{+\infty} d\mathbf{t}\ \mathbf{t}^{2s-1}\, e^{-\mathbf{t}\sqrt{\mathcal{A}}}(\mathbf{x},\mathbf{y})$$

(**holding also for $\mathbf{y}=\mathbf{x}\in\Omega$**; similar relations for $\partial^\alpha_{\mathbf{x}}\partial^\beta_{\mathbf{y}}\mathcal{A}^{-s}(\mathbf{x},\mathbf{y})$).

- Construct the **analytic continuation** of $\mathcal{A}^{-s}(\mathbf{x},\mathbf{y})$:

3) *Hankel representation*: if $\exists\, \mathcal{U}\subset\mathbb{C}$ with $[0,+\infty)\subset\mathcal{U}$, $H : [0,+\infty)\times\Omega\times\Omega \to \mathbb{R}$ s.t. $H(\ ;\mathbf{x},\mathbf{y}) : \mathcal{U}\to\mathbb{C}$ is *analytic* $(\mathbf{x},\mathbf{y}\in\Omega)$, $e^{-\mathbf{t}\sqrt{\mathcal{A}}}(\mathbf{x},\mathbf{y}) = \frac{1}{\mathbf{t}^d} H(\mathbf{t};\mathbf{x},\mathbf{y})$, then for all $s\in\mathbb{C}$ ($\mathfrak{H}$ = Hankel contour)

$$\mathcal{A}^{-s}(\mathbf{x},\mathbf{y}) = \frac{e^{-2i\pi s}\,\Gamma(1-2s)}{2i\pi}\int_{\mathfrak{H}} d\mathbf{t}\ \mathbf{t}^{2s-d-1}\, H(\mathbf{t};\mathbf{x},\mathbf{y})\,.$$

  - If $s=-k/2$, $k\in\mathbb{N}$, residue theorem $\Rightarrow$ *easy computation*.

3. Zeta-regularized scalar field.

- Canonical quantization:
  - $\mathfrak{F}^{\vee}(\mathcal{H}) := \bigoplus_{n=0}^{+\infty} \mathcal{H}^{\vee n} =$ **bosonic Fock space** on $\mathcal{H} := L^2(\Omega)$;
  - $\hat{a}^{\pm}(h)$ $(h \in \mathcal{H})$ = **creation/annihilation operators** on $\mathfrak{F}_0^{\vee}(\mathcal{H})$ ($\mathfrak{F}_0^{\vee}(\mathcal{H}) :=$ finite particle subspace; $[\hat{a}^-(h), \hat{a}^+(k)] \subset \langle h|k \rangle \mathbb{I}$; $\hat{a}^-(h)\mathbf{v} = \mathbf{0}$);
  - The **Wightman field at time zero** is

$$\hat{\varphi}(h) := \frac{1}{\sqrt{2}}\Big(\hat{a}^-(\,\overline{\mathcal{A}^{-1/4}h}\,) + \hat{a}^+(\,\mathcal{A}^{-1/4}h\,)\Big) \quad (h \in \mathcal{H}^{-1/2})\,.$$

- **Time evolution** via *second quantization* $(t \in \mathbb{R} = \text{time})$:

$$\hat{\varphi}_t(h) := \Gamma(e^{it\sqrt{\mathcal{A}}})\,\hat{\varphi}(h)\,\Gamma(e^{-it\sqrt{\mathcal{A}}}) \qquad (h \in \mathcal{H}^{-1/2})$$

  $\Rightarrow$ *Klein-Gordon eq.* : $(\partial_{tt}\hat{\varphi}_t(h) + \hat{\varphi}_t(\mathcal{A}\,h))\,\mathbf{f} = \mathbf{0}$ $(\mathbf{f} \in \mathfrak{F}_0^{\vee}(\mathcal{H}))$.

- $\forall\, \mathbf{x} \in \Omega$, $\delta_{\mathbf{x}} \notin \mathcal{H}^{-1/2}$ $(\delta_{\mathbf{x}} \in \mathcal{H}^{-r},\ r > d/2)$ $\Rightarrow$ pointwise evaluation of the field "$\hat{\varphi}(\mathbf{x}) := \hat{\varphi}(\delta_{\mathbf{x}})$" is ill-defined.

  $\hookrightarrow$ Basic idea: define a **regularized delta** $(\kappa \in \mathbb{R} = \textit{mass parameter})$

$$\delta_{\mathbf{x}}^u := (\mathcal{A}/\kappa^2)^{-u/4}\,\delta_{\mathbf{x}} \ \in\ \mathcal{H}^{-r} \quad \text{for } \Re u > d - 2r.$$

- The **zeta-regularized field** at a point $x = (t, \mathbf{x}) \in \mathbb{R} \times \Omega$ is

$$\hat{\varphi}^u(x) := \hat{\varphi}_t(\delta^u_{\mathbf{x}}) \qquad (\Re u > d-1)\,.$$

- The **zeta-regularized stress-energy tensor** is $(\mu, \nu \in \{0, ..., d\})$

$$\hat{T}^u_{\mu\nu}(x) := (1-2\xi)\,\partial_\mu \hat{\varphi}^u(x) \circ \partial_\nu \hat{\varphi}^u(x) +$$
$$+ \left(\frac{1}{2} - 2\xi\right)\left(\partial^\lambda \hat{\varphi}^u(x) \partial_\lambda \hat{\varphi}^u(x) + V(\mathbf{x})\,\hat{\varphi}^u(x)^2\right) - 2\xi \hat{\varphi}^u(x) \circ \partial_{\mu\nu} \hat{\varphi}^u(x)$$

  - well-defined for $\Re u > d+3$;
  - by analogy with classical theory ($\xi \in \mathbb{R}$ conformal parameter);
  - $A \circ B := (AB + BA)/2 \;\Rightarrow\; \hat{T}^u_{\mu\nu}(x) = \hat{T}^u_{\nu\mu}(x)$.

- The **zeta-reg. stress-energy VEV** is $\langle \mathbf{v} | \hat{T}^u_{\mu\nu}(x) \mathbf{v} \rangle \equiv \langle \mathbf{v} | \hat{T}^u_{\mu\nu}(\mathbf{x}) \mathbf{v} \rangle$
  - connection with the *Casimir effect*;
  - for $\Re u > n + d + 1$, the map $\Omega \ni \mathbf{x} \mapsto \langle \mathbf{v} | \hat{T}^u_{\mu\nu}(\mathbf{x})\, \mathbf{v} \rangle$ is C^n;
  - the properties of integral kernels give, e.g.,

$$\langle \mathbf{v} | \hat{T}^u_{00}(\mathbf{x})\, \mathbf{v} \rangle =$$
$$\kappa^u \left[\left(\tfrac{1}{4} + \xi\right) \mathcal{A}^{-\frac{u-1}{2}}(\mathbf{x}, \mathbf{y}) + \left(\tfrac{1}{4} - \xi\right) \left(\partial^{x^j} \partial_{y^j} + V(\mathbf{x})\right) \mathcal{A}^{-\frac{u+1}{2}}(\mathbf{x}, \mathbf{y}) \right]_{\mathbf{y}=\mathbf{x}}.$$

- The **renormalized VEV** is defined via analytic continuation ($\mathbf{x} \in \Omega$):

$$\langle \mathbf{v} | \hat{T}_{\mu\nu}(\mathbf{x})\, \mathbf{v} \rangle_{ren} := RP\Big|_{u=0} \langle \mathbf{v} | \hat{T}^{u}_{\mu\nu}(\mathbf{x})\, \mathbf{v} \rangle \;,$$

$RP|_{u=0} :=$ regular part of the Laurent expansion at $u = 0$.

N.B.: *Mellin relations* and the *asymptotic expansion of the heat kernel*
$\Rightarrow$ $\exists$ analytic continuation, *meromoprhic* near $u = 0$
$\Rightarrow$ $\forall\, \mathbf{x} \in \Omega$, $\langle \mathbf{v} | \hat{T}_{\mu\nu}(\mathbf{x})\, \mathbf{v} \rangle_{ren}$ is well-defined and finite.

- Related observables:
  - the **renormalized pressure** at $\mathbf{x} \in \partial\Omega$ is ($n(\mathbf{x}) =$ outer normal at $\mathbf{x}$)

$$p^{ren}_i(\mathbf{x}) := RP\Big|_{u=0} \langle \mathbf{v} | \hat{T}^{u}_{ij}(\mathbf{x})\, \mathbf{v} \rangle n^j(\mathbf{x}) \qquad (i, j = 1, ..., d)\,.$$

  - the **renormalized energy** is

$$\mathcal{E}^{ren} := RP|_{u=0} \int_\Omega d\mathbf{x}\, \langle \mathbf{v} | \hat{T}^{u}_{00}(\mathbf{x})\, \mathbf{v} \rangle \;.$$

4.1 The harmonic background potential.

- For simplicity: massless field, 3D isotropic harmonic potential
$$\Omega = \mathbb{R}^3, \quad V(\mathbf{x}) := \lambda^4 |\mathbf{x}|^2 \ (\lambda > 0) \quad \Rightarrow \quad \mathcal{A} = -\triangle + \lambda^4 |\mathbf{x}|^2$$
(*generalizations: massive field, anisotropic potentials, higher dimension*).
- Actor and Bender (1995): *total energy*, **not** the stress-energy tensor.
- **Computational methods**:
  - heat kernel: $e^{-t\mathcal{A}}(\mathbf{x}, \mathbf{y}) =$ Mehler kernel;
  - pass to *rescaled spherical coordinates*: $r := \lambda|\mathbf{x}| \in (0, +\infty)$;
  - the Mellin relations give (for $\Re u > 4$)
$$\langle \mathbf{v} | \hat{T}^u_{\mu\nu}(r)\, \mathbf{v} \rangle = \frac{\lambda^4}{\Gamma(\frac{u+1}{2})} \left(\frac{\kappa}{\lambda}\right)^u \int_0^{+\infty} d\tau\, \tau^{-3+\frac{u}{2}}\, H^{(u)}_{\mu\nu}(\tau\,; r)\,,$$
where $\tau \to H^{(u)}_{\mu\nu}(\tau; r)$ is smooth on $[0, +\infty)$;
  - 3-fold *integration by parts* $\Rightarrow$ analytic continuation to $\{\Re u > -2\}$
$$\langle \mathbf{v} | \hat{T}^u_{\mu\nu}(r)\, \mathbf{v} \rangle = -\frac{\lambda^4}{\Gamma(\frac{u+1}{2})(\frac{u}{2}-2)(\frac{u}{2}-1)\frac{u}{2}} \left(\frac{\kappa}{\lambda}\right)^u \int_0^{+\infty} d\tau\, \tau^{\frac{u}{2}}\, \partial^3_\tau H^{(u)}_{\mu\nu}(\tau\,; r)\,.$$

- The renormalized stress-energy VEV is $(\Diamond = \text{conf.}, \blacksquare = \text{non-conf.})$

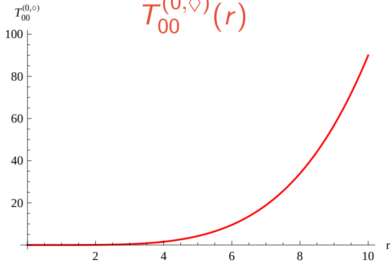
$$\langle \mathbf{v} | \hat{T}_{\mu\nu}(r) | \mathbf{v} \rangle_{ren} =$$

$$\lambda^4 \left[\left(T^{(0,\Diamond)}_{\mu\nu}(r) + M_{\kappa\lambda}\, T^{(1,\Diamond)}_{\mu\nu}(r) \right) + \left(\xi - \tfrac{1}{6} \right) \left(T^{(0,\blacksquare)}_{\mu\nu}(r) + M_{\kappa\lambda}\, T^{(1,\blacksquare)}_{\mu\nu}(r) \right) \right],$$

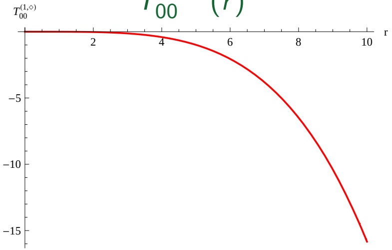
$$T^{(a,\bullet)}_{\mu\nu}(r) := \int_0^{+\infty} d\tau\, e^{-r^2 \tanh \tau} \left[\begin{array}{c} \text{polinomial in } r^2 \text{ with} \\ \text{coefficients depending on } \tau \end{array} \right], \qquad M_{\kappa\lambda} := \gamma_{EM} + 2 \ln\left(\frac{2\kappa}{\lambda} \right).$$

- Numerical evaluation of integral representations for fixed $r \in (0, +\infty)$:

$T^{(0,\Diamond)}_{00}(r)$



$T^{(1,\Diamond)}_{00}(r)$



- Small and large $r = \lambda |\mathbf{x}|$ asymptotics :
exact expressions and **explicit remainder estimates**.

- The renormalized stress-energy VEV is $(\lozenge = \text{conf.}, \blacksquare = \text{non-conf.})$

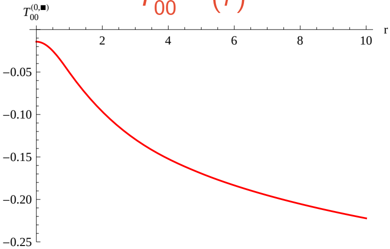
$$\langle \mathbf{v} | \hat{T}_{\mu\nu}(r) | \mathbf{v} \rangle_{ren} =$$

$$\lambda^4 \Big[\Big(T^{(0,\lozenge)}_{\mu\nu}(r) + M_{\kappa\lambda}\, T^{(1,\lozenge)}_{\mu\nu}(r) \Big) + \Big(\xi - \tfrac{1}{6} \Big) \Big(T^{(0,\blacksquare)}_{\mu\nu}(r) + M_{\kappa\lambda}\, T^{(1,\blacksquare)}_{\mu\nu}(r) \Big) \Big],$$

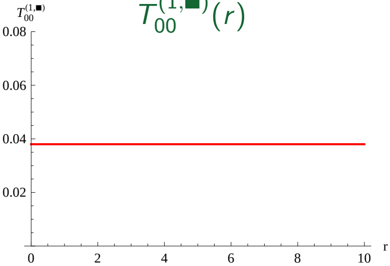
$$T^{(a,\bullet)}_{\mu\nu}(r) := \int_0^{+\infty} d\tau\, e^{-r^2 \tanh\tau} \left[\begin{array}{c} \text{polinomial in } r^2 \text{ with} \\ \text{coefficients depending on } \tau \end{array} \right], \quad M_{\kappa\lambda} := \gamma_{EM} + 2 \ln\left(\frac{2\kappa}{\lambda} \right).$$

- Numerical evaluation of integral representations for fixed $r \in (0, +\infty)$:

$T^{(0,\blacksquare)}_{00}(r)$



$T^{(1,\blacksquare)}_{00}(r)$



- Small and large $r = \lambda|\mathbf{x}|$ asymptotics :
exact expressions and **explicit remainder estimates**.

4.2 Parallel planes configuration: Robin b.c.

- Case study: massless field ($V = 0$), 3D model, Robin b.c.

$$\Omega = (0,a)\times\mathbb{R}^2 \ni (x^1,x^2,x^3) \quad (a>0)\,, \qquad \mathcal{A} = -\triangle$$
$$(1-\beta\partial_{x^1})\hat{\varphi}|_{x^1=0} = (1-\beta\partial_{x^1})\hat{\varphi}|_{x^1=a} = 0 \quad (\beta>0)\,.$$

- Romeo and Saharian (2002): *double* series/integral representation.

- **Computational methods** (like *Dirichlet*, *Neumann* and *periodic b.c.*):
  - reduced 1D problem on $(0,a)$:
    integral representation of the **cylinder kernel** $e^{-\mathbf{t}\sqrt{\mathcal{A}_1}}(x^1,y^1)$
    $\hookrightarrow$ $\forall\, x^1, y^1 \in (0,a)$, $[0,+\infty)\ni \mathbf{t} \mapsto e^{-\mathbf{t}\sqrt{\mathcal{A}_1}}(x^1,y^1)$ is *meromorphic* ($\mathbf{t}=0$ the only pole) and *decays exponentially* for $\mathbf{t}\to+\infty$.
  - **Hankel representations** of the Mellin relations, evaluated with the **residue theorem**, give, e.g.,

$$\langle \mathbf{v}|\,\hat{T}_{00}(x^1)\,\mathbf{v}\rangle_{ren} =$$
$$\frac{1}{\pi}\,\mathrm{Res}\left(\frac{1}{\mathbf{t}^4}\left[\left(3\xi-\frac{1}{4}\right)e^{-\mathbf{t}\sqrt{\mathcal{A}_1}} + \frac{\mathbf{t}^2}{2}\left(\frac{1}{4}-\xi\right)\partial_{x^1y^1}\,e^{-\mathbf{t}\sqrt{\mathcal{A}_1}}\right]_{y^1=x^1};\ \mathbf{t}=0\right).$$

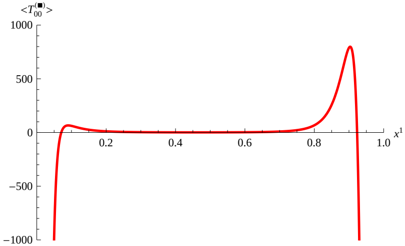
- The **renormalized stress-energy VEV**:
  - $\langle \mathbf{v} | \hat{T}_{\mu\nu} \mathbf{v} \rangle_{ren} = \langle T^{(\lozenge)}_{\mu\nu} \rangle + \left(\xi - \frac{1}{6}\right) \langle T^{(\blacksquare)}_{\mu\nu} \rangle$ (= conf. + non conf.) ;
  - diagonal $\rightarrow$ *non-vanishing components* :

$$\langle T^{(\lozenge)}_{00} \rangle = \frac{1}{3} \langle T^{(\lozenge)}_{11} \rangle = -\langle T^{(\lozenge)}_{22} \rangle = -\langle T^{(\lozenge)}_{33} \rangle = -\frac{\pi}{1440 a^4} \,,$$

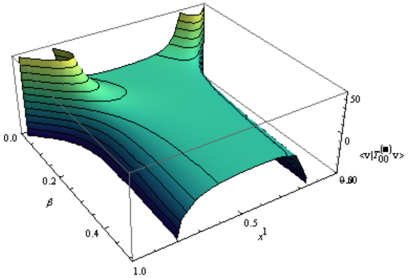
(like in the cases of Dirichlet or Neumann b.c.)

$$\langle T^{(\blacksquare)}_{00} \rangle = -\langle T^{(\blacksquare)}_{22} \rangle = -\langle T^{(\blacksquare)}_{33} \rangle = (\textit{single}) \text{ integral representation}$$

$\langle T^{(\blacksquare)}_{00} \rangle$ v.s. x^1 $(a = 1, \beta = 0.04)$



$\langle T^{(\blacksquare)}_{00} \rangle$ v.x. (x^1, β) $(a = 1)$



- It appears that $\int_{(0,a)} \langle \mathbf{v} | \hat{T}_{00} \mathbf{v} \rangle_{ren}$ diverges (but $E^{ren} < \infty \rightsquigarrow$ *anomaly*).

Summary and outlook.

- Summary:
  - abstract formalism to study integral kernels;
  - ZR in the framework of canonical quantization;
  - computational effectiveness in some examples.
- Further developments:
  - explicit analysis of other configurations;
  - study boundary divergences: semiclassical boundaries [Ford,1998];
  - functional-integral approach: regularized Gaussian measures.

Thank you for the attention!

Heuristic arguments for divergent expressions in QFT.

- **Canonical quantization** over a fixed spatial domain $\Omega \subset \mathbb{R}^d$ of the Klein-Gordon field: $(\partial_{tt} - \triangle)\hat{\varphi}(t,\mathbf{x}) = 0$, $(t,\mathbf{x}) \in \mathbb{R} \times \Omega$

  $\hookrightarrow$ creation/annihilation operators expansion ($\hat{a}_k|0\rangle = 0$, $\forall k \in \mathcal{K}$)

$$\hat{\varphi}(t,\mathbf{x}) = \int_{\mathcal{K}} \frac{dk}{\sqrt{2\omega_k}} \left[e^{-i\omega_k t} F_k(\mathbf{x})\, \hat{a}_k + e^{i\omega_k t} \bar{F}_k(\mathbf{x})\, \hat{a}_k^\dagger \right],$$

  $(F_k)_{k\in\mathcal{K}}$ Hilbert basis of $L^2(\Omega)$, $-\triangle F_k = \omega_k^2 F_k$, $[\hat{a}_h, \hat{a}_k^\dagger] = \delta_{hk}$;

- Computation of the **field squared VEV** gives a divergent sum over modes (formally related to the **integral kernel** of $(-\triangle)^{-\frac{1}{2}}$)

$$\langle 0|\hat{\varphi}(t,\mathbf{x})^2|0\rangle = \int_{\mathcal{K}} \frac{dk}{2\,\omega_k} F_k(\mathbf{x})\bar{F}_k(\mathbf{x}) \ \left(\text{“=”}\ \frac{1}{2} \langle \delta_{\mathbf{x}}|(-\triangle)^{-\frac{1}{2}} \delta_{\mathbf{x}}\rangle \right).$$

▲ **Example**: $\Omega := (0,1)$ with Dirichlet b.c. $\Rightarrow$
$F_k(x) = \sqrt{2}\sin(k\pi x)$, $\omega_k = k\pi$ for $k = 1, 2, 3, \ldots \Rightarrow$

$$\langle 0|\hat{\varphi}(t,\mathbf{x})^2|0\rangle = \sum_{k=1}^{+\infty} \frac{\sin^2(k\pi x)}{k\pi}\,.$$

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