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Gibbs Measures for the Periodic Derivative Nonlinear Schrödinger Equation

a joint work with
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Bressanone, 08/02/2016

1. The DNLS equation

The DNLS equation:

$$i\psi_t = \psi_{xx} + i\beta \left(\psi|\psi|^2\right)_x \ .$$

If $\psi(x,t)$ is a solution then

$$\psi_a(x,t) := a^\alpha \psi(ax, a^2 t)$$

is also a solution iff $\alpha = 1/2$.

	periodic	**non periodic**
LWP	Herr 06 in $H^{s\geq 1/2}$, Herr, Grünrock 08 in $\mathcal{F}L^{s,r}$ $r \in (2,4),\ s \geq 1/2$	Takaoka 99 in $H^{s\geq 1/2}$
GWP	Win 10, $H^{s>1/2}(\mathbb{T})$	Miao,Wu,Xu 11 $H^{s\geq 1/2}(\mathbb{R})$.

All these results hold for small data in L^2.

The energy

$$H_1 := \frac{1}{2}\|\psi\|^2_{\dot{H}^1} + \frac{3i}{4}\beta\int \psi^2\bar{\psi}\bar{\psi}' + \frac{\beta^2}{4}\|\psi\|^6_{L^6}\ ,$$

is bounded from below for

$$\|\psi\|_{L^2} \leq \frac{1}{\sqrt{K|\beta|}},$$

where K is the optimal constant in the Gagliardo-Nirenberg inequality

$$\|u\|^3_{L^6} \leq K\|u\|_{\dot{H}^1}\|u\|^2_{L^2} + K'\|u\|^3_{L^2}\ .$$

$K' = 0$ on $\mathbb{R}$. $K_{\mathbb{R}} = \sqrt{2\pi}$ while $K_{\mathbb{T}}$ is unknown. The critical mass is $K/\sqrt{|\beta|}$, which saturates GN. This is the *ground state* mass.

Recently: GWP in H^1 for data with mass exceeding the ground state (Wu, '14; Oh-Monsicat '15).

DNSL is an integrable PDE (Kaup-Newell '66; De Sole-Kac '13), i.e. there are infinitely many functionals conserved by the flow (for regular enough solutions). Some of them follows:

$$\int h_0 = \frac{1}{2}\|\psi\|_{L^2}^2 \,, \quad \textbf{(mass)}$$

$$\int h_1 = \frac{i}{2}\int \psi\bar{\psi}' + \frac{\beta}{4}\|\psi\|_{L^4}^4 \,, \quad \textbf{(momentum)}$$

$$\int h_2 = \frac{1}{2}\|\psi\|_{\dot{H}^1}^2 + \frac{3i}{4}\beta\int \psi^2\bar{\psi}\bar{\psi}' + \frac{\beta^2}{4}\|\psi\|_{L^6}^6 \,, \quad \textbf{(energy)}$$

$$\int h_3 = \frac{i}{2}\int \psi'\bar{\psi}'' + \frac{\beta}{4}\int \left((\psi')^2\bar{\psi}^2 + 8\psi\bar{\psi}\psi'\bar{\psi}' + \psi^2(\bar{\psi}')^2\right) + \frac{5i}{4}\beta^2\int \psi^3\bar{\psi}^2\bar{\psi}' + \frac{5}{16}\beta^3\|\psi\|_{L^8}^8 \,,$$

$$\int h_4 = \frac{1}{2}\|\psi\|_{\dot{H}^2}^2 + \frac{5i}{4}\beta\int \left(\psi\bar{\psi}\psi'\bar{\psi}'' - \psi\bar{\psi}\psi''\bar{\psi}'\right) + \frac{5}{4}\beta^2\int \left(\psi\bar{\psi}^3(\psi')^2 + 5\psi^2\bar{\psi}^2\psi'\bar{\psi}' + \psi^3\bar{\psi}(\bar{\psi}')^2\right) + \frac{35i}{16}\beta^3\int \psi^4\bar{\psi}^3\bar{\psi}' + \frac{7}{16}\beta^4\|\psi\|_{L^{10}}^{10} \,.$$

2. Gibbs measures for PDEs

Gaussian measure on $L^2(\mathbb{T})$:

$$(\mathbb{I} - \Delta^k)e_n = (1 + n^{2k})e_n \quad \text{self-adjoint, the inverse is trace class.}$$

All functions $u(x) \in L^2(\mathbb{T})$ can be written in Fourier series:

$$u(x) = \sum_{n \in \mathbb{Z}} u_n e_n \, .$$

Define

$$\gamma_k^N(A) := \frac{\prod_{|n| \leq N} \sqrt{1 + n^{2k}}}{(2\pi)^{2N+1}} \int_A du_{-N} d\bar{u}_{-N} ... du_N d\bar{u}_N e^{-\frac{1}{2} \sum_{|n| \leq N} (1+n^{2k})|u_n|^2}$$

complex Gaussian measure of $A \subseteq \mathbb{C}^{2N+1}$. For $B \in \mathscr{B}(\mathbb{C}^{2N+1})$

$$M_N(B) = \{u \in L^2(\mathbb{T}) \mid [(u, e_{-N}), \ldots, (u, e_N)] \in B\} \quad \text{cylindrical set} \, .$$

Extension to $L^2(\mathbb{T})$ by $\gamma_k(M_N) := \gamma_k^N(M_N)$ and Kolmogorov reconstruction.

Some remarkable properties:

- The measure is concentrated on functions in $L^2(\mathbb{T})$ having slightly less then $k - \frac{1}{2}$ weak derivatives as regularity:

$$\forall k \geq 0\,, \qquad \gamma_k \left(\bigcap_{\varepsilon > 0} H^{k - \frac{1}{2} - \varepsilon} \right) = 1\,.$$

- Hypercontractivity: for f a r-linear function we have

$$\|f\|_{L^p(\gamma_k)} \leq (p-1)^{r/2} \|f\|_{L^2(\gamma_k)}\,.$$

the L^2 norm controls the $L^{p \geq 2}$ norms.

Consider the NL Schrödinger eq. (on $\mathbb{T}$)

$$i\partial_t \psi = \Delta\psi + Q[\psi, \bar{\psi}]$$

and the conserved quantities $\|\psi\|_{L_2}$ and

$$H[\psi, \bar{\psi}] = \int \bar{\psi}(\Delta\psi + Q[\psi, \bar{\psi}]).$$

A natural Gibbs measure associated to this system is

$$Z^{-1} e^{\int \bar{\psi} Q[\psi,\bar{\psi}]} e^{-\|\psi\|^2_{\dot{H}_1}} d[\psi, \bar{\psi}] = Z^{-1} e^{\int \bar{\psi} Q[\psi,\bar{\psi}]} \gamma_1(\psi)\,,$$

where Z is a normalisation factor. This is equivalent to consider *a system of Gaussian spins* with interaction

$$\int \bar{\psi} Q[\psi, \bar{\psi}]\,.$$

Then we can use the spherical constraint $\|\psi\|_{L_2} = 1$ to have a more precise measure:

$$\bar{Z}^{-1} e^{\int \bar{\psi} Q[\psi,\bar{\psi}]} 1_{\{\|\psi\|_{L_2}=1\}} \gamma_1(\psi)\,.$$

A bit of history on Gibbs measures for PDEs:

- Lebowitz, Rose, Speer 88: general idea, construction of the measure for NLS equations on $\mathbb{T}$ (subcritical and critical);

- Bourgain 94: Invariance for the measure of LRS;

- Zhidkov 91-00: Costruction of invariant Gibbs measures for KdV and NLS equation;

- Bourgain 94-02: Costruction of invariant Gibbs measures for several equation: NLS on line, defocusing NLS in 2D, Gross-Pitaevski...

- Bridges, Slade 96: Costruction of the Gibbs measures for focusing NLS eq. in 2D;

- Mc Kean, Vanisvski 94-96: Costructuon of invariant Gibbs measures for NLS, KdV, Klein Gordon and Sine Gordon;

more recently

- Tzvetkov, Visciglia, Deng 08-14: Construction of invariant Gibbs measures for Benjamin-Ono equation;

- Thomann, Tzetvkov 10: Construction of the Gibbs measure (associated to the energy) of DNLS equation;

- Nahmod, Oh, Rey-Bellet, Sheffield, Staffilani 11-12: Construction of the invariant Gibbs measure (associated to the energy) of DNLS equation;

many others in the PDEs community in the last years...

3. The main theorem

$\chi_R : \mathbb{R} \to [0,1]$ cut-off function. For $k \geq 2$:

$$\textstyle\int h_0 \simeq R_0, \ldots \int h_{k-1} \simeq R_{k-1}.$$

The density

$$G_{k,N}(\psi) := \Big(\prod_{m=0}^{k-1} \chi_{R_m} \left(\textstyle\int h_{2m} \right) \Big) e^{-\int q_k(\psi_N)} .$$

The associated measure $d\rho_{k,N}$ is

$$\rho_{k,N}(d\psi) = G_{k,N}(\psi)\gamma_k(d\psi) .$$

Theorem (G., Lucà, Valeri 15)**.** *Let* $k \geq 2$, $R_0\sqrt{|\beta|} \ll 1$. *Then* $G_{k,N}(\psi) \xrightarrow{\gamma_k} G_k(\psi)$, *as* $N \to \infty$. *Moreover, there exists*

$$p_0(R_0, R_1, \ldots, R_{k-1}, k, |\beta|) > 1$$

such that for $p \in [1, p_0)$ $G_k(\psi) \in L^p(\gamma_k)$ *and* $G_{k,N}(\psi) \xrightarrow{L^p(d\gamma_k)} G_k(\psi)$.

Note:

- $G_k(u)$ is supported on a set of positive measure w.r.t. $\gamma_k(u)$;

- The measure $\rho_{k,N}$ weakly converges, as $N \to \infty$, to the (Gibbs) measure ρ_k on $L^2(\mathbb{T})$:

$$\rho_k(du) = G_k(u)\gamma_k(du)\,;$$

- ρ_k is $k-1$ times micro-canonical and one time canonical;

- the construction is valid for any finite $k \geq 2$, but we cannot control the limit $k \to \infty$;

- the condition $p_0 > 1$ requires the smallness assumption (either on $|\beta|$ or on the L_2 norm R_0).

4. Strategy of the proof

0. Analysis of the integrals of motions. Via the *Lenard-Magri* scheme De Sole and Kac were able to write a recursive relation for the integrals of motions.

Proposition 1 (G., Lucà, Valeri 15) *For every* $n \in \mathbb{Z}_+$ *the conserved quantities* $\int h_{2n}$ *for the DNLS have the form:*

$$\int h_{2n} = \frac{1}{2}\int \psi^{(n)}\bar{\psi}^{(n)} + \underbrace{\frac{(2n+1)i}{2}\beta\int \bar{\psi}^{(n)}\psi^{(n-1)}\bar{\psi}\psi}_{:= \int q_n} + \int R_{2n} \, ,$$

where R_{2n} *is a polynomial in* $\psi^{(k)}, \bar{\psi}^{(k)}$ *of total differential degree strictly less than* $2n-1$.

The idea (roughly):

$$\int R_{2n}[\psi] \lesssim \|\psi\|_{H^{n-1}} ,$$

which lies in the support of γ_n.

So we have isolated the bad term to control.

1. Control of the Sobolev norm with the integrals of motion:

Proposition. *Let $k \in \mathbb{Z}_+$. For every $0 \leq m \leq k$ let us fix $R_m \geq 0$, and let us assume that $R_0 \ll 1/\sqrt{|\beta|}$. There exists $\mathcal{C} = \mathcal{C}(R_0, \ldots, R_k, k, |\beta|)$ such that if*

$$|\int h_{2m}[\psi]| \leq R_m, \quad \textit{for any} \quad m = 0, \ldots, k\,,$$

then

$$\|\psi\|_{\dot{H}^k} \leq \mathcal{C}.$$

2. Convergence of the integrals of motion for $N \to \infty$

Proposition. *Let $k \geq 2$ and $1 \leq m \leq k$. Then $\int q_m(\psi_N)$ converges in measure to $\int q_m(\psi)$ w.r.t. the Gaussian measure $d\gamma_k$. Furthermore, if $1 \leq m < k$, then $\int h_{2m}(\psi_N)$ converges in measure to $\int h_{2m}(\psi)$ w.r.t. $d\gamma_k$.*

2.a. Convergence is point-wise for all the terms but $f_N^k := \int \bar{\psi}_N^{(k)} \psi_N^{(k-1)} \bar{\psi}_N \psi_N$

2.b For f_N^k we have

Proposition. *Let $k \geq 2$. For all $N > M \geq 1$, we have*

$$\|f_M^k - f_N^k\|_{L^2(\gamma_k)} \lesssim \frac{1}{\sqrt{M}} \, ,$$

thereby $\forall p > 2$

$$\|f_M^k(\psi) - f_N^k(\psi)\|_{L^p(\gamma_k)} \lesssim \frac{(p-1)^2}{\sqrt{M}} \, .$$

This is a crucial estimate. It is proved by passing to Fourier modes and then exploiting the **Wick contractions** induced by the Gaussian measure.

Note: at this point we already proved convergence in probability.

3. The final step of our analysis the probabilistic estimate

Proposition. *Take* $R_0 \leq \sqrt{\frac{2}{9|\beta|}}$ *such that*

$$p_0 := \min\left(2\left(3(2k+1)|\beta|\sqrt{\mathcal{C}R_0^3}\right)^{-1}, \left(4(2k+1)|\beta|R_0\mathcal{C}\right)^{-1}\right) > 1.$$

Then for any $k \geq 2$, $p \in [1, p_0)$ *and* $N \geq \left(\frac{2k+1}{2}|\beta|\right)^2 R_0^6 \mathcal{C}^2$

$$\|G_{k,N}(\psi)\|_{L^p(\gamma_k)} \leq C < +\infty\,.$$

end of the proof:

γ_k-conv. + $L_p(\gamma_k)$-integrability yields $L_p(\gamma_k)$-conv. of $G_{N,k}$

Outlooks.

- extend to the real line, $\sqrt{|\beta|}R_0$ small; (*challenging*)
- prove the invariance w.r.t. the Schrödinger flow, $\sqrt{|\beta|}R_0$ small; (*much challenging*)
- go beyond the regime $\sqrt{|\beta|}R_0$ small. (*very much challenging*)

Thank you for the attention!