GRAN SASSO SCIENCE INSTITUTE
CENTER FOR ADVANCED STUDIES
Istituto Nazionale di Fisica Nucleare

Quantum Hydrodynamic Systems and applications to superfluidity at finite temperatures

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Superfluidity

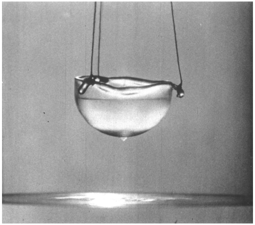

Figure: Superfluid Helium ^{4}He

source: Alfred Leitner - superfluid liquid Helium

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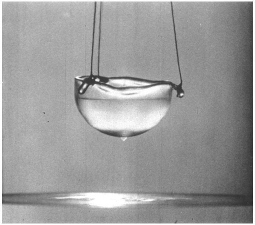

Figure: Superfluid Helium ^{4}He

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- frictionless flow through narrow capillarities;
- irrotational outside the nodal region, $\nabla \wedge v = 0$ in $\{\rho > 0\}$;
- quantized vortices: $m \oint_C v \cdot dl = 2\pi\hbar n, \; n \in \mathbb{Z}$.

Quantum Hydrodynamics (QHD)

$$\begin{cases} \partial_t \rho + \operatorname{div} J = 0 \\ \partial_t J + \operatorname{div}\left(\frac{J \otimes J}{\rho}\right) + \nabla P(\rho) = \frac{\hbar^2}{2} \rho \nabla \left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right), \end{cases}$$

with $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^3$, and initial data $\rho(0) = \rho_0, J(0) = J_0$.
Mass (charge) density ρ, momentum (current) density J.

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$$\frac{1}{2}\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) = \frac{1}{4}\mathrm{div}(\rho\nabla^2\log\rho) = \frac{1}{4}\nabla\Delta\rho - \mathrm{div}(\nabla\sqrt{\rho}\otimes\nabla\sqrt{\rho}).$$

Finite energy weak solutions

$$\begin{cases} \partial_t \rho + \operatorname{div} J = 0 \\ \partial_t J + \operatorname{div}\left(\frac{J \otimes J}{\rho}\right) + \nabla P(\rho) = \frac{\hbar^2}{2}\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right), \end{cases}$$

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$$\text{Energy: } E[\rho, J] = \int \frac{\hbar^2}{2}|\nabla\sqrt{\rho}|^2 + \frac{1}{2}\frac{|J|^2}{\rho} + f(\rho)\,dx,$$

$$P(\rho) = \rho f'(\rho) - f(\rho) = \frac{\gamma - 1}{\gamma}\rho^\gamma, \quad 1 \leq \gamma < 3.$$

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Aim: find $(\sqrt{\rho}, \Lambda)$ such that $\rho := (\sqrt{\rho})^2, J := \sqrt{\rho}\Lambda$ is a finite energy weak solution.

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Global existence of F.E.W.S.

Theorem

For any $\psi_0 \in H^1(\mathbb{R}^3)$*, let* $\rho_0 := |\psi_0|^2, J_0 := \hbar \operatorname{Im}(\bar{\psi}_0 \nabla \psi_0)$*, then there exist* $(\sqrt{\rho}, \Lambda)$ *such that* (ρ, J) *is a finite energy weak solution for the QHD system.*

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- *No uniqueness!*

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Analogy with NLS - WKB

$$\begin{cases} i\hbar\partial_t\psi = -\frac{\hbar^2}{2m}\Delta\psi + f'(|\psi|^2)\psi \\ \psi(0) = \psi_0. \end{cases}$$

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WKB ansatz: $\psi = \sqrt{\rho}e^{iS/\hbar}$, then (ρ, S) satisfy

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$$\hbar^2 |\nabla \psi|^2 = \hbar^2 |\nabla \sqrt{\rho}|^2 + \rho |\nabla S|^2 = \hbar^2 |\nabla \sqrt{\rho}|^2 + \frac{|J|^2}{\rho}$$

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- vacuum: WKB ansatz only valid when $\psi(t, x) \neq 0 \rightsquigarrow S$ not defined in $\{\psi = 0\}$
- regularity issue: the set $\{\psi = 0\}$ when $\psi \in H^1(\mathbb{R}^3)$ may be irregular (Federer, Ziemer)
- irrotationality: $\nabla \wedge u = 0$, no vortices are taken into account in the WKB description

Polar Factorization

$\forall\ \psi \in H^1(\mathbb{R}^3)$, define

$$P(\psi) = \left\{\phi \in L^\infty \text{ s.t. } \|\phi\|_{L^\infty} \leq 1, \psi = |\psi|\phi \text{ a.e. } \mathbb{R}^3\right\}.$$

$\forall\ \phi \in P(\psi)$, then $|\phi| = 1\ \sqrt{\rho}\, dx$-a.e. in $\mathbb{R}^3$ and it is uniquely defined $\sqrt{\rho}\, dx-$a.e. in $\mathbb{R}^3$.
Not uniquely defined in $\{\rho = 0\}$!
We call (any) $\phi \in P(\psi)$ polar factor associated to ψ.

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Polar factorisation: $\psi = \sqrt{\rho}\phi$, $\rightsquigarrow$ (QHD)

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Lemma (Polar factorisation)

Let $\phi \in L^\infty(\mathbb{R}^3)$ *be such that* $\psi = |\psi|\phi$ *a.e. and* $\|\phi\|_{L^\infty(\mathbb{R}^3)} \leq 1$. *Then*

$$\nabla\sqrt{\rho} = \mathrm{Re}(\bar{\phi}\nabla\psi) \text{ a.e.}, \quad \Lambda := \hbar\mathrm{Im}(\bar{\phi}\nabla\psi) \text{ a.e.}$$

and

$$\hbar^2\mathrm{Re}(\nabla\bar{\psi} \otimes \nabla\psi) = \hbar^2\nabla\sqrt{\rho} \otimes \nabla\sqrt{\rho} + \Lambda \otimes \Lambda \quad \text{a.e.}$$

Furthermore the decomposition is H^1-*stable, i.e. if* $\{\psi_n\} \subset H^1(\mathbb{R}^3)$ *s.t.* $\psi_n \to \psi$ *in* H^1, *then*

$$\nabla\sqrt{\rho_n} \to \nabla\sqrt{\rho}, \quad \Lambda_n \to \Lambda \; in \; L^2(\mathbb{R}^3).$$

Remark

- *In general we only have* $\phi_n \rightharpoonup \phi$ *weak*$-*$ *in* L^∞.
- $\forall\, \psi \in W^{1,1}_{loc}(\mathbb{R}^3)$, $\nabla\psi = 0$ *a.e. in* $\psi^{-1}(\{0\})$.

QHD through Polar Fact. (Madelung transformation)

Moments associated to the wave function. ψ soln. to NLS
$\rho := |\psi|^2$ *mass density*

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Rigorously: density argument, continuity of NLS w.r.t. initial data, H^1 stability of polar factorisation.

Energy and generalized irrotationality condition

$$E[\sqrt{\rho}, \Lambda] = \int \frac{\hbar^2}{2}|\nabla\sqrt{\rho}|^2 + \frac{1}{2}|\Lambda|^2 + f(\rho)\,dx$$

Finite energy weak solutions to (QHD) satisfy

$$\nabla \wedge J = 2\nabla\sqrt{\rho} \wedge \Lambda, \quad \text{for a.e. t.}$$

If the solutions are smooth (e.g. the velocity field can be defined), then the generalized irrotationality condition is equivalent to

$$\rho\nabla \wedge u = 0, \quad \text{a.e.}$$

i.e. the velocity field is irrotational $\rho\,dx$ a.e.

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QHD with collisions

$$\begin{cases} \partial_t \rho + \operatorname{div} J = 0 \\ \partial_t J + \operatorname{div}\left(\frac{J \otimes J}{\rho}\right) + \nabla P(\rho) + \rho \nabla V + J = \frac{\hbar^2}{2}\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) \\ -\Delta V = \rho \end{cases}$$

Momentum relaxation term introduced to phenomenologically model collisions between electrons in the semiconductor device (Bløtekjær, Baccarani, Wordeman).

Energy:

$$E[\rho, J] = \int_{\mathbb{R}^3} \frac{\hbar^2}{2}|\nabla\sqrt{\rho}|^2 + \frac{1}{2}\frac{|J|^2}{\rho} + f(\rho) + \frac{1}{2}|\nabla V|^2 \, dx,$$

dissipates along the flow of solutions

$$E(t) + \int_0^t \int_{\mathbb{R}^3} \frac{|J|^2}{\rho} \, dx dt' = E(0).$$

The collision term J "destroys" the analogy with NLS,

$$\begin{cases} i\partial_t\psi = -\frac{1}{2}\Delta\psi + V\psi + f'(|\psi|^2)\psi + \arg(\psi)\psi \\ -\Delta V = |\psi|^2 \end{cases}$$

No good Cauchy theory for this equation.
Previous results for QHD with collisions:

- [Jüngel, Mariani Rial, M3AS, 2002]: local existence for regular initial data, bounded away from zero.
- [Li, Marcati, CMP, 2004]: local regular solutions, global "subsonic" solutions.

Theorem

Let $\psi_0 \in H^1(\mathbb{R}^3)$ *and let* $\rho_0 := |\psi_0|^2, J_0 := \hbar \mathrm{Im}(\bar{\psi}_0 \nabla \psi_0)$. *Then there exists a global in time finite energy weak solution* (ρ, J) *to the QHD system with collisions with initial data* (ρ_0, J_0). *The solution satisfies*

$$\sqrt{\rho} \in L^\infty(\mathbb{R}_+ : H^1(\mathbb{R}^3)), \Lambda \in L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^3)) \cap L^2(\mathbb{R}_+; L^2(\mathbb{R}^3))$$

and

$$\sqrt{\rho} \in L^q([0, T]; W^{1,r}(\mathbb{R}^3)), \Lambda \in L^q([0, T]; L^r(\mathbb{R}^3)),$$

for any $0 < T < \infty$, *where* (q, r) *is any arbitrary (Strichartz) admissible pair for Schrödinger in* $\mathbb{R}^3$.

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Strategy of proof

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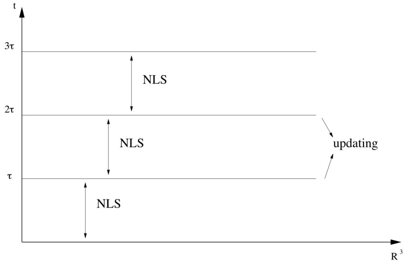
Difficulties

- "Good" definition of approximate solutions;
- Prove sufficient a priori estimates to get the compactness.

Fractional step

- solve the QHD without collisions (NLS)
- update with collisions



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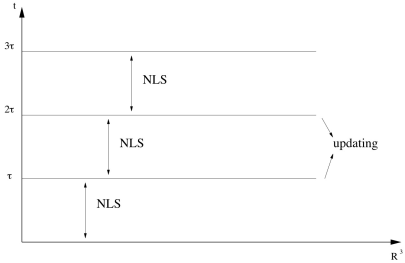

$$\partial_t J + J = 0$$

Fractional step

- solve the QHD without collisions (NLS)
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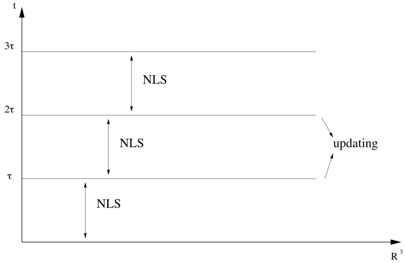

$$\partial_t J + J = 0 \Rightarrow J_{new} = e^{-\tau} J_{old} \sim (1-\tau) J_{old}$$

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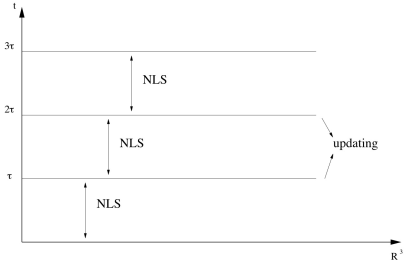

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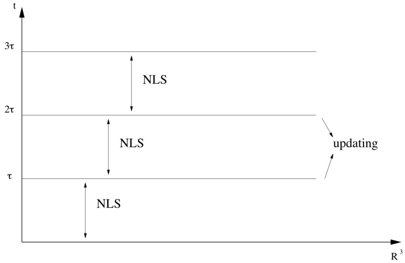

$$\partial_t J + J = 0 \Rightarrow J_{new} = e^{-\tau} J_{old} \sim (1-\tau) J_{old}$$

$$S_{new} = (1-\tau) S_{old}, \quad \psi_{new} = (\phi_{old})^{1-\tau} \sqrt{\rho_{old}}$$

More exactly...

Lemma (Updating)

Let $\psi_o \in H^1(\mathbb{R}^3)$ *and let* $\varepsilon, \tau > 0$ *be two arbitrary, small real numbers. Then there exists* $\psi_n \in H^1(\mathbb{R}^3)$ *s.t.*

$$\rho_n = \rho_o, \quad \Lambda_n = (1-\tau)\Lambda_o + r_\varepsilon,$$

where

$$\|r_\varepsilon\|_{L^2} \leq \varepsilon,$$

and

$$\nabla\psi_n = \nabla\psi_o - i\frac{\tau}{\hbar}\phi^*\Lambda_o + r_{\varepsilon,\tau},$$

with

$$\|\phi^*\|_{L^\infty} \leq 1, \quad \|r_{\varepsilon,\tau}\|_{L^2} \leq C(\tau\|\nabla\psi_o\|_{L^2} + \varepsilon).$$

Consistency of the approximate solutions

Proposition

Let us assume there exist $\sqrt{\rho} \in L^2_{loc}([0,T); H^1_{loc}(\mathbb{R}^3)), \Lambda \in L^2_{loc}([0,T); L^2_{loc}(\mathbb{R}^3))$, $0 < T < \infty$ *such that*

$$\sqrt{\rho^\tau} \to \sqrt{\rho} \quad in \; L^2_{loc}([0,T) : H^1_{loc}(\mathbb{R}^3))$$
$$\Lambda^\tau \to \Lambda \quad in \; L^2_{loc}([0,T); L^2_{loc}(\mathbb{R}^3)),$$

where $(\sqrt{\rho^\tau}, \Lambda^\tau)$ *is the sequence of approximate solutions constructed above. Then* $(\sqrt{\rho}, \Lambda)$ *defines a finite energy weak solution to the QHD system with collisions.*

Compactness of approximate solutions

With the fractional step we get a sequence $\{\psi^\tau\}$ (and consequently $\{(\sqrt{\rho^\tau}, \Lambda^\tau)\}$). Energy dissipation for approximate solutions:

$$E^\tau(t) + \frac{\tau}{2} \sum_{k=0}^{[t/\tau]-1} \int |\Lambda^\tau(k\tau-)|^2 \, dx \leq (1+\tau) E_0.$$

Up to passing to subsequences,

$$\psi^\tau \rightharpoonup \psi \quad \text{in } L^\infty(\mathbb{R}_+ : H^1(\mathbb{R}^3)).$$

Not sufficient for the quadratic terms

$$\nabla\sqrt{\rho^\tau} \otimes \nabla\sqrt{\rho^\tau} + \Lambda^\tau \otimes \Lambda^\tau = \mathrm{Re}(\nabla\bar{\psi}^\tau \otimes \nabla\psi^\tau)$$

Need further compactness: Use dispersion

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Dispersive estimates for Schrödinger semigroup

Strichartz estimates (Strichartz, Ginibre-Velo, Keel-Tao)
(q, r) are *admissible* if $2 \leq q \leq \infty, 2 \leq r \leq 6$ and $\frac{1}{q} = \frac{3}{2}\left(\frac{1}{2} - \frac{1}{r}\right)$

$$\|e^{\frac{i}{2}t\Delta}f\|_{L^q_t L^r_x} \lesssim \|f\|_{L^2}$$

$$\|\int_0^t e^{\frac{i}{2}(t-s)\Delta}F(s)\,ds\|_{L^q_t L^r_x} \lesssim \|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x}$$

Local smoothing estimates (Constantin-Saut, Sjölin, Vega)

$$\|e^{\frac{i}{2}t\Delta}f\|_{L^2([0,T];H^{1/2}_{loc}(\mathbb{R}^3))} \lesssim \|f\|_{L^2}$$

$$\|\int_0^t e^{\frac{i}{2}(t-s)\Delta}F(s)\,ds\|_{L^2([0,T];H^{1/2}_{loc}(\mathbb{R}^3))} \lesssim \|F\|_{L^1_t L^2_x}$$

Compactness estimates

From the updating Lemma

$$
\begin{aligned}
\nabla\psi^\tau(t) =& e^{i\frac{t}{2}\Delta}\nabla\psi_0 - \frac{i}{\hbar}\int_0^t e^{i\frac{(t-s)}{2}\Delta}\nabla(V^\tau\psi^\tau + f'(|\psi^\tau|^2)\psi^\tau)(s)\,ds \\
& - i\frac{\tau}{\hbar}\sum_{k=0}^{[t/\tau]-1} e^{i\frac{(t-k\tau)}{2}\Delta}\left(\phi^*_{k,\tau}\Lambda^\tau(k\tau-) + r_{k,\tau}\right).
\end{aligned}
$$

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Compactness estimates

From the updating Lemma

$$\nabla\psi^\tau(t) = e^{i\frac{t}{2}\Delta}\nabla\psi_0 - \frac{i}{\hbar}\int_0^t e^{i\frac{(t-s)}{2}\Delta}\nabla(V^\tau\psi^\tau + f'(|\psi^\tau|^2)\psi^\tau)(s)\,ds$$
$$- i\frac{\tau}{\hbar}\sum_{k=0}^{[t/\tau]-1} e^{i\frac{(t-k\tau)}{2}\Delta}\left(\phi^*_{k,\tau}\Lambda^\tau(k\tau-) + r_{k,\tau}\right).$$

By Strichartz estimates (with a standard bootstrap argument)

$$\|\nabla\psi^\tau\|_{L^q_t L^r_x([0,T]\times\mathbb{R}^3)} \leq C(E_0, M_0, T)$$

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Compactness estimates

From the updating Lemma

$$\nabla\psi^\tau(t) = e^{i\frac{t}{2}\Delta}\nabla\psi_0 - \frac{i}{\hbar}\int_0^t e^{i\frac{(t-s)}{2}\Delta}\nabla(V^\tau\psi^\tau + f'(|\psi^\tau|^2)\psi^\tau)(s)\,ds$$
$$- i\frac{\tau}{\hbar}\sum_{k=0}^{[t/\tau]-1} e^{i\frac{(t-k\tau)}{2}\Delta}\left(\phi^*_{k,\tau}\Lambda^\tau(k\tau-) + r_{k,\tau}\right).$$

By Strichartz estimates (with a standard bootstrap argument)

$$\|\nabla\psi^\tau\|_{L^q_t L^r_x([0,T]\times\mathbb{R}^3)} \le C(E_0, M_0, T)$$

and by using this + local smoothing

$$\|\nabla\psi^\tau\|_{L^2([0,T]:H^{1/2}_{loc}(\mathbb{R}^3))} \le C(E_0, M_0, T),$$

for any $0 < T < \infty$.

Aubin-Lions type lemma

Proposition

For any $0 < T < \infty$,

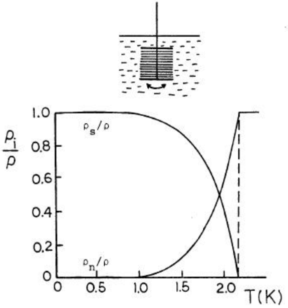
$$\nabla\psi^\tau \to \nabla\psi \quad in \ L^2([0,T]; L^2_{loc}(\mathbb{R}^3)),$$

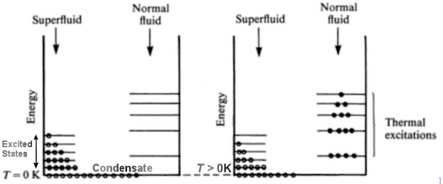
up to passing to subsequences. In particular,

$$\nabla\sqrt{\rho^\tau} \to \nabla\sqrt{\rho},\ \Lambda^\tau \to \Lambda \quad in \ L^2([0,T]; L^2_{loc}(\mathbb{R}^3)).$$

By the consistency of the sequence of approximate solutions, $(\rho, J) := ((\sqrt{\rho})^2, \sqrt{\rho}\Lambda)$ is a finite energy weak solution to the QHD system with collisions in $[0,T] \times \mathbb{R}^3$, for any $0 < T < \infty$.
Thus the global existence theorem follows.

Superfluidity at finite temperatures (work in progress with P. Marcati, M. D'Amico)





Helium II State

Landau two fluid model

Khalatnikov, *An introduction to the theory of Superfluidity*
Griffin, Nikuni, Zaremba, *Bose-condensed gases at finite temperatures*

$$
\begin{cases}
\partial_t \rho_s + \operatorname{div}(\rho_s v_s) = -\Gamma_{12} \\
\partial_t (\rho_s v_s) + \operatorname{div}(\rho_s v_s \otimes v_s) \\
\qquad + \nabla P_s(\rho_s) + \rho_s \nabla V_{ext} = \frac{1}{2}\rho_s \nabla \left(\frac{\Delta \sqrt{\rho_s}}{\sqrt{\rho_s}} \right) - Q_{12} \\
\partial_t \rho_n + \operatorname{div}(\rho_n v_n) = -\Gamma_{21} \\
\partial_t (\rho_n v_n) + \operatorname{div}(\rho_n v_n \otimes v_n) \\
\qquad + \nabla P_n(\rho_n) + \rho_n \nabla V_{ext} = \operatorname{div}\left(2\eta \left(\mathrm{D}\, v_n - \frac{1}{3} \mathbf{1}\, \mathrm{Tr}\, \mathrm{D}\, v_n \right) \right) - Q_{21} \\
\quad \text{entropy eqn.}
\end{cases}
$$

Superfluidity near the $\lambda-$point/BEC at finite temperatures.

Landau two fluid model - (very) simplified

$V_{ext} = 0$, $\Gamma_{12} = \Gamma_{21} = 0$, $Q_{21} = 0$, $Q_{12} = \frac{1}{\tau}\rho_1(v_1 - v_2)$.

Landau two fluid model - (very) simplified

$V_{ext} = 0$, $\Gamma_{12} = \Gamma_{21} = 0$, $Q_{21} = 0$, $Q_{12} = \frac{1}{\tau}\rho_1(v_1 - \mathbb{Q}v_2)$.

$$\begin{cases} \partial_t \rho_1 + \operatorname{div} J_1 = 0 \\ \partial_t J_1 + \operatorname{div}\left(\frac{J_1 \otimes J_1}{\rho_1}\right) + \nabla P_1(\rho_1) = \frac{1}{2}\rho_1 \nabla \left(\frac{\Delta\sqrt{\rho_1}}{\sqrt{\rho_1}}\right) - \frac{1}{\tau}(J_1 - \rho_1 \mathbb{Q} v_2) \\ \partial_t \rho_2 + \operatorname{div}(\rho_2 v_2) = 0 \\ \partial_t(\rho_2 v_2) + \operatorname{div}(\rho_2 v_2 \otimes v_2) + \nabla P_2(\rho_2) = \eta \Delta v_2 + \frac{1}{3}\eta \nabla \operatorname{div} v_2, \end{cases}$$

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$$\int \frac{1}{2}\rho_2 |v_2|^2 + f_2(\rho_2)\, dx + \eta \int_0^t \int |\nabla v_2|^2 + \frac{1}{3}|\operatorname{div} v_2|^3\, dx \leq \int \frac{1}{2}\rho_{2,0}|v_{2,0}|^2 + f_2(\rho_{2,0})\, dx.$$

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Superfluid part at NLS level

$$i\partial_t\psi = -\frac{1}{2}\Delta\psi + \tilde{V}\psi + f_1'(|\psi|^2)\psi, \tag{1}$$

where $\tilde{V}$ s.t. $\nabla\tilde{V} = -\mathbb{Q}v_2$.

Theorem (Ortner, Süli, 2012)

$\tilde{V} = V_\infty + V_p$, where

- *for a.e. $t \in \mathbb{R}$, $V_\infty(t) \in C^\infty(\mathbb{R}^3)$;*
- *$V_p \in L^2_t W^{1,6}_x$ and $\|V_p\|_{L^2_t W^{1,6}_x} \leq \|\nabla\tilde{V}\|_{L^2_t L^6_x} \lesssim \|v_2\|_{L^2_t L^6_x}$;*
- *$\|\partial^\alpha V_\infty\|_{L^2_t L^\infty_x} \leq C\|\nabla\tilde{V}\|_{L^2_t L^6_x}$, for all $|\alpha| \geq 1$.*

Ingredients: GWP for (1), Strichartz and local smoothing estimates for (1), fractional step.

Construction of the fundamental solution of the Schrödinger equation

Theorem (Fujiwara, J. Anal. Math. 1979)

Assume

- $\forall\, t \in \mathbb{R},\ V(t,\cdot) \in C^\infty(\mathbb{R}^d)$;
- $\forall\, \alpha \in \mathbb{N}^d, |\alpha| \geq 2,\ \sup_{(t,x)\in\mathbb{R}\times\mathbb{R}^d} |\nabla^\alpha V(t,x)| \leq C.$

Then there exists unitary operator $U(t,s)$ such that $U(t,s)f$ is the solution to

$$\begin{cases} i\partial_t u = -\frac{1}{2}\Delta u + Vu \\ u(s) = f. \end{cases}$$

Problem: V^∞ is such that $\nabla^\alpha V \in L^2_t L^\infty_x(\mathbb{R}\times\mathbb{R}^d)$! (work in progress)

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Towards a more realistic two-fluids model...

Equation for the order parameter (Khalatnikov)

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2m}\Delta\psi + f'(|\psi|^2)\psi - i\lambda m\left[\frac{1}{2}\left(-\frac{i\hbar}{m}\nabla - v_n\right)^2\psi + f_1'(|\psi|^2)\psi\right]$$

Total mass and momentum density evolutions

$$\begin{cases} \partial_t(\rho_s + \rho_n) + \operatorname{div}(\rho_s v_s + \rho_n v_n) = 0 \\ \partial_t(\rho_s v_s + \rho_n v_n) + \operatorname{div}(\rho_s v_s \otimes v_s + \rho_n v_n \otimes v_n + p\mathbb{I}) = \\ \qquad \operatorname{div}\left(\eta\, \mathrm{D}\, v_n - \frac{2}{3}\eta \operatorname{div} v_n \mathbb{I}\right) \end{cases}$$

+entropy

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Conclusions

- existence of finite energy weak solutions for quantum fluids models: no further regularity or smallness assumptions;
- no need to define the velocity field: polar factorisation $\rightsquigarrow (\sqrt{\rho}, \Lambda)$;
- vacuum $\rightsquigarrow$ quantized vortices;

Future perspectives

- uniqueness/stability;
- (more physical) two-fluid models;
- quantum plasma physics (Quantum MHD).

References

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A. Marcati, *The Quantum Hydrodynamic System in two Space Dimensions*, ARMA 2012

A., Marcati, *Finite Energy Global Solutions to a Two-Fluid Model Arising in Superfluidity*, Bull. Acad. Sinica 2015.