SUB-RIEMANNIAN SANTALÓ FORMULA AND APPLICATIONS

with D. Prandi and L. Rizzi

Marcello Seri

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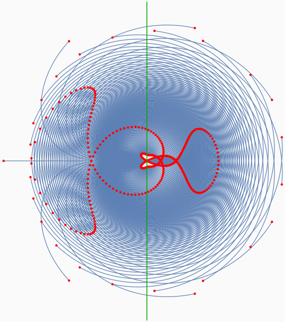
University of Reading

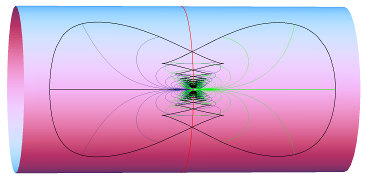
OUTLINE



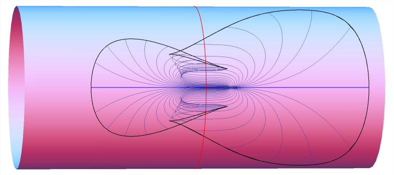
- Motivations
- Classical Santaló formula and applications
- Pills of Sub-Riemannian geometry
- Main result and applications
- Conclusions

Understand singular geometries that present **classical tunnelling** and **quantum confinement**

But most of the usual “tools” are missing



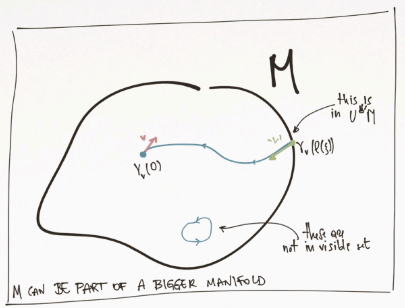
(a) geodesics starting from the singular set (red circle)



(b) geodesics starting from the the point (.3,0)

Figure 1: Geodesics for Grushin cylinder; red singular set; black wave front

(M, g) n-dimensional compact Riemannian manifold, $\partial M \neq \emptyset$ and smooth, $d\omega$ Riemannian volume, n_q $(q \in \partial M)$ unit inward normal.



- $UM = \{v \in TM \mid \|v\| = 1\}$ unit tangent bundle
- $\phi_t : UM \to UM$ geodesic flow, geodesic curve $\gamma_v(t) : M \to M$
- $\ell(v) \in [0, +\infty]$ exit time of the geodesic
- $U^{\text{☀}}M = \{v \in UM \mid \ell(-v) < +\infty\}$ visible unit tangent bundle

Let $F : UM \to \mathbb{R}$ be a measurable function. Then

$$\int_{U^{\Sunrise}M} F\, d\mu = \int_{\partial M} d\sigma(q) \int_{U_q^+\partial M} d\eta(v) \int_0^{\ell(v)} dt\, F(\phi_t(v))\, g(v, n_q)$$

where

- $d\mu$ is the *Liouville measure*;
- $d\sigma(q)$ is the surface measure on ∂M;
- $d\eta(v)$ is an opportune measure on

$$U_q^+\partial M := \{v \in U_q M \mid q \in \partial M, g(v, n_q) > 0\}.$$

Santaló formula implies the following inequalities [Croke and Derdzinski 1980, 1984, 1987].

- Hardy-Poincaré-like inequality: for $f \in C_0^\infty(M)$,

$$\int_M |\nabla f|^2 \geq \frac{n\pi^2}{L^2} \int_M |f|^2,$$

  where L is the lenght of the longest geodesic in M;
- Isoperimetric-like inequality:

$$\frac{\sigma(\partial M)}{\omega(M)} \geq \frac{2\pi|\mathbb{S}^{k-1}|}{|\mathbb{S}^k|} \frac{\theta^{\sunset}}{\mathrm{diam}(M)};$$

- few others, but you got the idea…

$\Rightarrow$ **powerful tool for geometric inequalities in a geometric way**

Classic sub-Riemannian manifold: (M, D, g) where

- M is an n-dimensional smooth manifold;
- $D \subset TM$ of rank $k \geq n$ is a family of k-dimensional linear subspaces $D_q \subseteq T_qM$ depending smoothly on $q \in M$;
- $g : D \to \mathbb{R}$ smooth function whose restrictions g_q to D_q are positive definite quadratic forms.

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$$\Rightarrow$$

- *horizontal curves* $\gamma(t)$: $\dot{\gamma}(t) \in D_{\gamma(t)}$ a.e.
- $length(\gamma) = \int_0^T \|\dot{\gamma}(t)\|_g \, dt$ induces distance d
- well defined *horizontal gradient* (∇_H) and *divergence*
- *Laplace-Beltrami* $-\Delta f = \mathrm{div}(\nabla_H f)$ on $L^2(M, g)$

The inner product is defined only on D, and there is no canonical way to extend it on $TM \Rightarrow$ work on the **cotangent space**

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Under an additional **invariance condition** on D, one can perform a **reduction** and get rid of all the unbounded directions.

Theorem (Reduced Santaló Formula)

Let $F : T^*M \to \mathbb{R}$ measurable. Then,

$$\int_{U^{\bullet}M^{\mathrm{r}}} F \, d\mu^{\mathrm{r}} = \int_{\partial M} d\sigma(q) \int_{U_q^*\partial M^{\mathrm{r}}} d\eta^{\mathrm{r}}(\xi) \int_0^{\ell(\xi)} dt \, F(\varphi_t(\xi)) \, \langle \xi, n_q \rangle$$

Like the classical formula if you replace (UM, ϕ_t) with (U^*M, φ_t) where $U^*M = \{\xi \in T^*M \mid H(\xi) = \frac{1}{2}\}$ and $\varphi_t : U^*M \to U^*M$. The little r stands for “after reduction”.

Also redefine $n_q \in T^*\partial M$ and $U^{\bullet}M \subseteq U^*M$

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The additional condition holds for at least the following: submersions, contact geometries, systems with symmetries, CR geometries, quaternionic (real and complex) and octonionic fibrations, any Carnot group

With the previous formula one can immediately show the following. Let $f \in C_0^\infty(M)$.

- **Hardy-like inequality**:

$$\int_M |\nabla_H f|^2 d\omega \geq \frac{k\pi^2}{|\mathbb{S}^{k-1}|} \int_M \frac{f^2}{R^2}\, d\omega$$

  where $\frac{1}{R^2} = \int_{U_q^* M^{\mathrm{r}}} \frac{1}{L^2(\xi)} d\eta^{\mathrm{r}}(\xi)$;
- p-**Hardy inequalities**, $p > 1$ (omitted);
- **Poincaré-like inequality**: $\int_M |\nabla_H f|^2 d\omega \geq \frac{k\pi^2}{L^2} \int_M f^2\, d\omega$;
- **Isoperimetric-like inequality** (omitted);
- **Bound on the first Dirichlet eigenvalue**: $\lambda_1(M) \geq \frac{k\pi^2}{L^2}$

FOLLOW-UP

Possible to use above inequalities to get prove **quantum trapping by volume explosion** (also in cases with classical tunnelling), however the proof has severe restrictions on the dimension of the distribution.

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Also, if D is the principal bundle, the Laplace-Beltrami is the usual magnetic Laplacian.

THANK YOU FOR THE ATTENTION!