Topological edge states in two-gap unitary systems: a transfer matrix approach

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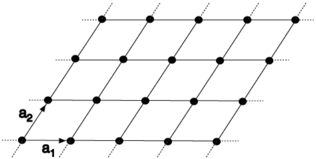
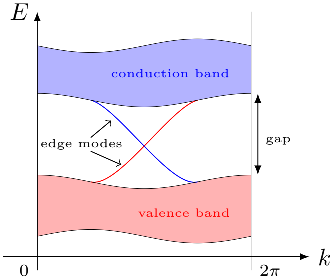
joint work with P. Delplace and M. Fruchart (LPENSL)

New Journal of Physics 17 (2015) 115008

Monday, February 8th 2016

Motivations





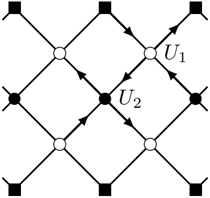
- Context: topological insulators and band theory.
- Hermitian system + translation invariance = Bloch bands and bulk topological invariants (e.g. first Chern number)
- On a cylinder geometry: topologically protected edge modes

What about systems ruled by unitary operators ?

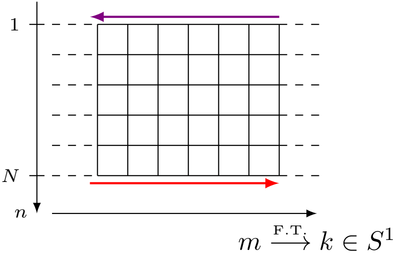
e.g : periodically driven (Floquet) systems, scattering processes,...

One specific model at $d = 2$

- Scattering of light [Pasek, Chong, '14] on an oriented lattice [Ho-,Chalker,-Coddington '96 '88]] in a cylinder geometry.



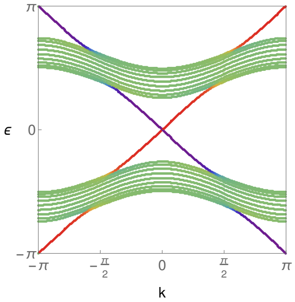
$U_1, U_2 \in \mathcal{U}(2)$



- State $\psi(k) \in \mathbb{C}^{2N}$ ruled by

$$U(k) \equiv \begin{pmatrix} 1 & & \\ & \tilde{U}_2(k) \otimes I_{N-1} & \\ & & 1 \end{pmatrix} \cdot \left(\tilde{U}_1(k) \otimes I_N \right) \in \mathcal{U}(2N)$$

The spectrum



$$U\psi = \mathrm{e}^{-\mathrm{i}\epsilon}\psi$$

- Periodic spectrum in k and ϵ
- Bulk bands: delocalized states
- Localized edge modes in both gaps

Bulk invariants (1st Chern numbers) are all vanishing,
but there are still topologically protected edge modes !

How to characterize them ?

Reformulation

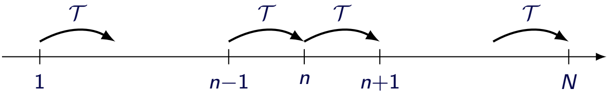
$$U(k) = \begin{pmatrix} \alpha & \beta & & & & \\ \lceil & 2\times 4 & \rceil & & & \\ \lfloor & & \rfloor & & & \\ & & & \ddots & & \\ & & & & \lceil \; 2\times 4 & \rceil \\ & & & & \lfloor & \rfloor \\ & & & & \delta & \gamma \end{pmatrix}, \qquad \psi = \begin{pmatrix} A_1 \\ B_1 \\ \dots \\ A_n \\ B_n \\ \dots \\ A_N \\ B_N \end{pmatrix}$$

Eigenvalue problem $U\psi = \mathrm{e}^{-\mathrm{i}\epsilon}\psi$ can be reconstructed from transfer matrix

$$\begin{pmatrix} A_{n+1} \\ B_{n+1} \end{pmatrix} = \mathcal{T}(k,\epsilon) \begin{pmatrix} A_n \\ B_n \end{pmatrix} \qquad \mathcal{T} \in \mathcal{M}_2(\mathbb{C})$$

and some boundary constraints at $n = 1$ (and similarly for N).

$$\begin{pmatrix} A_1 \\ B_1 \end{pmatrix} \in \mathcal{D}_1(k,\epsilon;\alpha,\beta) \cong \mathbb{C}^1$$



At a given (k, ϵ):

- $\det \mathcal{T} = 1$ and $Sp(\mathcal{T}) = \{\lambda_+, \lambda_- = 1/\lambda_+\} \in \mathbb{C}^2$
- $|\lambda_+| = |\lambda_-| = 1 \quad \Rightarrow \quad |(A_{n+1}, B_{n+1})| \sim |(A_n, B_n)|$
  Delocalized bulk state
- $|\lambda_+| > 1$, $|\lambda_-| < 1 \quad \Rightarrow \quad \mathcal{T}^n v_+ = (\lambda_+)^n v_+$

Edge modes exponentially localized at $n = 1$ (resp. N) as eigenstates associated to λ_- (resp. λ_+) of the transfer matrix.

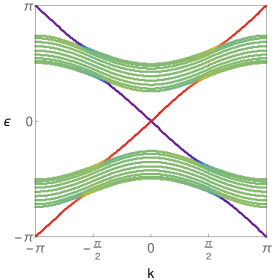
Solving

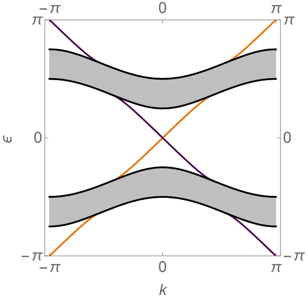
Initial eigenvalue problem for U reduced to the eigenvalue problem of $\mathcal{T}$:

$$\lambda^2 + 2f\lambda + 1 = 0 \qquad \text{with} \qquad f(k,\epsilon) = -1/2\,\mathrm{tr}\mathcal{T} \in \mathbb{R}$$

$$\Rightarrow \lambda = -f + \mu \qquad \text{where} \qquad \mu^2 = f^2 - 1$$

- $\mu \in \mathrm{i}\mathbb{R}$ for $f^2 < 1$: bulk bands
- $\mu \in \mathbb{R}$ for $f^2 > 1$: gaps
- Boundary constrains $\cap$ Eigenvectors of $\mathcal{T}$: edge modes (localization = sign of μ)



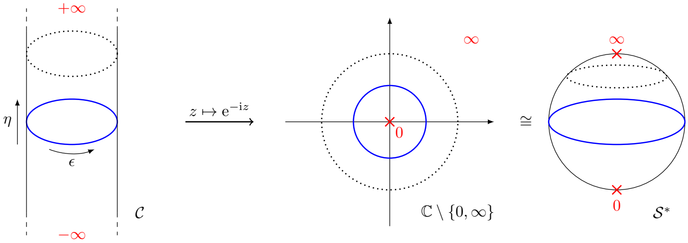


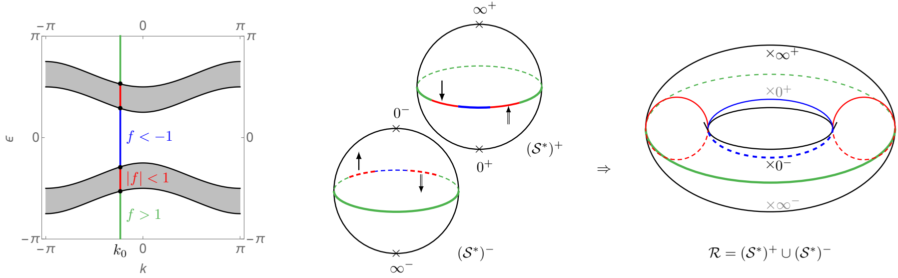
Analytic continuation

$\mathcal{T}$ allows to reconstruct each edge state of U both for eigenvalues $\epsilon(k)$ and eigenstates $\psi(k)$.

- Associated topological invariant ?
- Quantities involved : $P[e^{\pm ik}, e^{\pm i\epsilon}]$
- Singularity of $\mu^2 = f^2 - 1$ at $f = \pm 1$ (band/gap frontier)

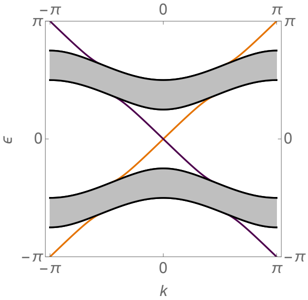
Analytic continuation of $\epsilon \mapsto z \equiv \epsilon + i\eta$

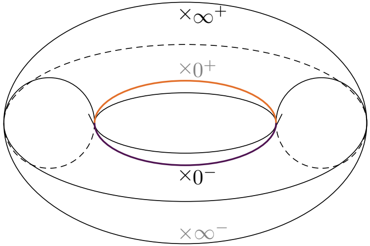




- For fixed $k = k_0 \to$ elliptic curve associated to $\mu^2 = f^2 - 1$:
  4 poles • where $f = \pm 1$
  2 branch cuts — where $\mu^2 < 0 \leftrightarrow |f| < 1$
- μ is analytic on $\mathcal{R}$ = gluing of $(S^*)^+$ and $(S^*)^-$ (punctured torus)
- The punctures ensure the distinction between the two gaps !
- One equivalent surface $\mathcal{R}_k$ for each k

Winding number

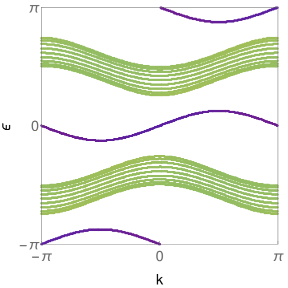


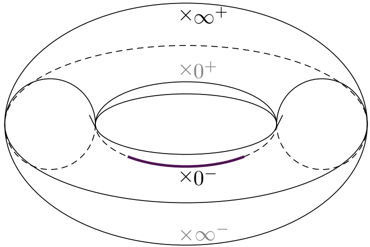


- Identify all the surfaces $\mathcal{R}_k$ to one single punctured torus $\mathcal{R}_0$ (flat family)
- On $\mathcal{R}_0$, the edge modes of a given gap wind (or not) around a non-contractible loop.

Toplogical invariant of a gap : winding number of the edge modes

Winding number





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Toplogical invariant of a gap : winding number of the edge modes

Conclusion and perspectives

Topological invariant associated to each gap of a unitary system, as winding number of the edge modes on a family of Riemann surfaces.

Adapts [Hatsugai '93] to the unitary case (extra singularities).

- Generalization to other unitary systems. Towards a bulk-edge correspondence ?
  [Graf, Porta '13], [Avila, Schulz-Baldes, Villegas-Blas '13]
- Symmetry class of the problem ? [Lein's Talk]
- Underlying dynamics and relation to Floquet systems.
  [Rudner *et al.* '12]

Thank you.