Self-adjointness and domain of the Fröhlich Hamiltonian

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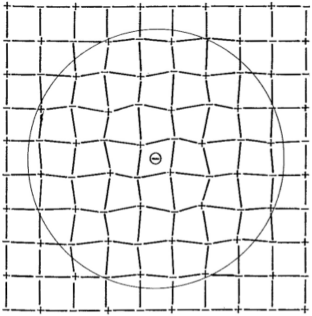
Physical background



An electron in an ionic crystal polarizes its surroundings by Coulomb interaction. Electron and lattice polarization (deformation) together constitute a quasi-particle, the so-called polaron. (Source: Madelung [1])

The Polaron is described by the *formal* Fröhlich Hamiltonian:
(see Devreese [2] or Feynman [3])

$$p^2 + \int a_k^* a_k dk + \sqrt{\alpha} \int \frac{1}{|k|} \left(e^{ikx} a_k + e^{-ikx} a_k^* \right) dk$$

Problem: No well-defined operator since $e^{\pm ikx}/|k| \notin L^2(\mathbb{R}^3)$.

$\rightarrow$ Introduction of an UV-cutoff

Mathematical description

The system is described by a state Ψ in the Hilbert space

$$\mathscr{H} = L^2(\mathbb{R}^d) \otimes \mathcal{F}\left(L^2(\mathbb{R}^d)\right)$$

with the symmetric Fock space

$$\mathcal{F}(L^2(\mathbb{R}^d)) := \bigoplus_{n\geq 0} \bigotimes_s^n L^2(\mathbb{R}^d).$$

For all $\Phi, \Psi \in \mathscr{H}$, we have the inner product

$$\langle \Phi, \Psi \rangle := \sum_{n=0}^{\infty} \int dx \int dk_1 \cdots \int dk_n \overline{\Phi^{(n)}(x; k_1, ..., k_n)} \Psi^{(n)}(x; k_1, ..., k_n).$$

Number and ladder operators

The number operator $N : D(N) \to \mathcal{F}$ is defined by

$$(N\Psi)^{(n)} := n\Psi^{(n)}.$$

For $f \in L^2(\mathbb{R}^d)$, we define the annihilation and creation operators on $D(\sqrt{N})$ by

$$\begin{aligned} a(f)\Psi^{(n)} &:= \sqrt{n}\,\left\langle f, \Psi^{(n)} \right\rangle, \\ a^*(f)\Psi^{(n)} &:= \sqrt{n+1}\; S_{n+1}\,\left(f \otimes \Psi^{(n)} \right). \end{aligned}$$

CCR: $$[a(f), a^*(g)] = \langle f, g \rangle \quad \text{on } D(N)$$

Fröhlich Hamiltonian with cutoff

We define the Hamiltonian with cutoff $\Lambda < \infty$ by

$$H_\Lambda := \underbrace{p^2 \otimes \mathbb{1}_{\mathcal{F}} + \mathbb{1}_{L^2} \otimes N}_{=H_0} + \phi(G_\Lambda)$$

on $D(H_0) := D(p^2) \cap D(N)$. Thereby, $p := -i\nabla$ and

$$\begin{aligned} \phi(G_\Lambda) &:= a(G_\Lambda) + a^*(G_\Lambda), \\ G_\Lambda(x,k) &:= |k|^{-(d-1)/2} e^{-ikx} \chi_\Lambda(k), \quad d \geq 2. \\ &(\to \|G_\infty\| = \infty) \end{aligned}$$

For all $\Lambda < \infty$, H_Λ is self-adjoint on $D(H_0)$ and bounded from below.

Self-adjoint realization of H (without cutoff)

1. as the self-adjoint operator associated to a quadratic form,
2. as the norm-resolvent limit of H_Λ,
3. as the generator of a strongly continuous unitary group,
4. by using the Gross Transformation.

$\rightarrow$ All approaches yield to the same unique self-adjoint operator H with $D(H) \subset D(H_0^{1/2})$.

$\rightarrow$ Only the Gross Transformation gives us an explicit representation of H and $D(H)$.

Gross transformation

We introduce for all $0 < \sigma < \Lambda \leq \infty$ the unitary operator

$$U_{\sigma,\Lambda} := e^{i\pi(B_{\sigma,\Lambda})}$$

with

$$\begin{aligned} i\pi(B_{\sigma,\Lambda}) &:= a(B_{\sigma,\Lambda}) - a^*(B_{\sigma,\Lambda}), \\ B_{\sigma,\Lambda}(x,k) &:= -\frac{1}{1+k^2} G_\Lambda(x,k)\Big(1 - \chi_\sigma(k)\Big), \\ G_\Lambda(x,k) &:= |k|^{-(d-1)/2} e^{-ikx} \chi_\Lambda(k), \quad d \geq 2. \end{aligned}$$

Note, that $\quad \|B_{\sigma,\infty}\| < \infty$, $\|kB_{\sigma,\infty}\| < \infty$, but $\left\|k^2 B_{\sigma,\infty}\right\| = \infty$.

We want to transform

$$U_{\sigma,\Lambda} H_\Lambda U^*_{\sigma,\Lambda} = U_{\sigma,\Lambda}\left(p^2 + N + \phi(G_\Lambda)\right) U^*_{\sigma,\Lambda}.$$

Self-adjointness of the transformed Hamiltonian

On $D(H_0)$ for $\Lambda < \infty$

$$\begin{aligned} U_{\sigma,\Lambda} H_\Lambda U^*_{\sigma,\Lambda} =& H_\sigma - 2a^*(kB_{\sigma,\Lambda}) \cdot p - 2p \cdot a(kB_{\sigma,\Lambda}) + a(kB_{\sigma,\Lambda})^2 \\ &+ a^*(kB_{\sigma,\Lambda})^2 + 2a^*(kB_{\sigma,\Lambda}) a(kB_{\sigma,\Lambda}) + C_{\sigma,\Lambda}. \end{aligned}$$

We define $H'_{\sigma,\Lambda}$ by the righthand side of this equation (even for $\Lambda = \infty$).

Theorem (Griesemer, Wünsch)

Let $a := a(kB_{\sigma,\infty})$ and $a^* := a^*(kB_{\sigma,\infty})$. For $\sigma > 0$ large enough we get the representation $H = U^*_{\sigma,\infty} H'_{\sigma,\infty} U_{\sigma,\infty}$ where

$$H'_{\sigma,\infty} := H_\sigma - 2a^* \cdot p - 2p \cdot a + a^2 + (a^*)^2 + 2a^* a + C_\sigma$$

is a self-adjoint operator on $D(H_0)$. It follows that

$$D(H) = U^*_{\sigma,\infty} D(H_0).$$

If $\mathcal{D} \subset D(H_0)$ is a core of H_0 then $U^*_{\sigma,\infty} \mathcal{D}$ is a core of H.

What is $D(H) = U^*_{\sigma,\infty} D(H_0)$?

We know the following mapping properties:

- for $\Lambda \leq \infty$: $U^*_{\sigma,\Lambda} D(H_0^{1/2}) = D(H_0^{1/2})$ $\quad \Rightarrow \quad$ $D(H) \subset D(H_0^{1/2})$
- for $\Lambda \leq \infty$: $U^*_{\sigma,\Lambda} D(N) = D(N)$
- for $\Lambda < \infty$: $U^*_{\sigma,\Lambda} D(H_0) \subset D(p^2)$

$\Rightarrow$ only for $\Lambda < \infty$: $U^*_{\sigma,\Lambda} D(H_0) = D(H_0)$

In fact, we have

$$U^*_{\sigma,\infty} D(H_0) \cap D(H_0) = \{0\}$$

which we get from the more general

Theorem (Griesemer, Wünsch)

$$(i) \quad D(H) \subset \left(\bigcap_{0 \leq s < 3/2} D(|p|^s) \right) \cap D(N)$$

$$(ii) \quad D(H) \cap D(|p|^{3/2}) = \{0\}$$

Idea of the proof

Theorem (Griesemer, Wünsch)

$$
\begin{aligned}
&(i) \quad D(H) \subset \left(\bigcap_{0 \le s < 3/2} D(|p|^s) \right) \cap D(N) \\
&(ii) \quad D(H) \cap D(|p|^{3/2}) = \{0\}
\end{aligned}
$$

Idea: $D(H) = U^*_{\sigma,\infty} D(H_0), \quad D(H_0) = D(p^2) \cap D(N)$

(i) Let $1 \le s \le 2$ and $\Psi \in D(H_0)$ and look when $\| |p|^s U^*_{\sigma,\infty} \Psi \| < \infty$.

(ii) Let $\Psi \in D(H_0)$ and suppose that $U^*_{\sigma,\infty} \Psi \in D(|p|^{3/2})$. Considering $\| |p|^{3/2} U^*_{\sigma,\infty} \Psi \|$ which then has to be finite, the only possibility is $\Psi = 0$.

Heuristic stuff

Remind:

- $D(H) = U^*_{\sigma,\infty} D(H_0)$
- $D(H) \cap D(H_0) = \{0\}$
- $U^*_{\sigma,\infty}$ maps cores of H_0 to cores of H

Consider for Ψ from a suitable domain in $\mathscr{H} \setminus \{0\}$

$$\Big(p^2 + \underbrace{N + a(G_\infty)}_{OK} + a^*(G_\infty)\Big)\Psi.$$

- $a^*(G_\infty)$ maps outside of $\mathscr{H}$ $\Rightarrow$ Ψ must not be $\in D(H_0)$.
- The set $\mathcal{D} := \{\varphi \otimes e^{i\pi(f)}\Omega \mid \varphi, f \in C_0^\infty(\mathbb{R}^d)\}$ is a core of H_0, thus $U^*_{\sigma,\infty}\mathcal{D}$ is a core of H. For $\Psi \in U^*_{\sigma,\infty}\mathcal{D}$, one can *formally* calculate the state above and see the compensation of the divergent terms.

$\Rightarrow$ Meaning of the *formal* expression above for no $\Psi \in D(H_0)$ except $\Psi = 0$.

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