Classification of Topological Insulators for Classical Light

in collarboration with Giuseppe De Nittis

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Talk Based on

Collaboration with **Giuseppe De Nittis**

- *On the Role of Symmetries in the Theory of Photonic Crystals*
  Annals of Physics **350**, pp. 568–587, 2014
- *On the Role of Symmetries and Topology in the Theory of Classical Electromagnetism*
  in preparation, 2016

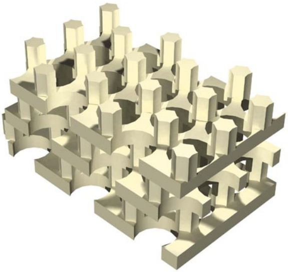
Motivation

Realizing Quantum Effects with Classical Light

Periodic Light Conductors

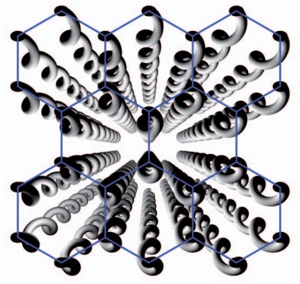
Photonic Crystals

Johnson & Joannopoulos (2004)



Periodic Waveguide Arrays

Rechtsman, Szameit et al (2013)



- periodic structure $\Longrightarrow$ *peculiar light conduction properties*
- artificial PLCs can be *engineered arbitrarily* and *inexpensively*
- **"band structure"** and **"band topology engineering"**
  $\rightsquigarrow$ **photonic band gaps**, slow light, low-dispersion materials

A Novel Class of Materials: *Photonic Topological Insulators*

Theory

Predicted by

- Raghu and Haldane (2005)

Experiment

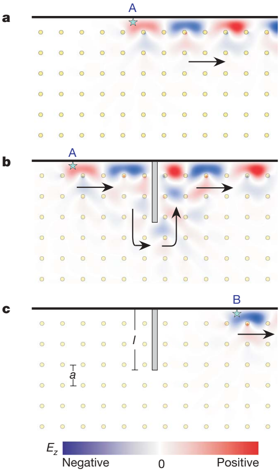
... and realized in

- 2d **photonic crystals** for **microwaves** by Joannopoulos, Soljačić et al (2009)
- **periodic waveguide arrays** for light at **optical frequencies** by Rechtsman, Szameit et al (2013)

A Novel Class of Materials: *Photonic Topological Insulators*

$$\left.\begin{pmatrix} \bar{\varepsilon} & 0 \\ 0 & \bar{\mu} \end{pmatrix} \neq \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix}\right\} \Longrightarrow$$

symmetry breaking



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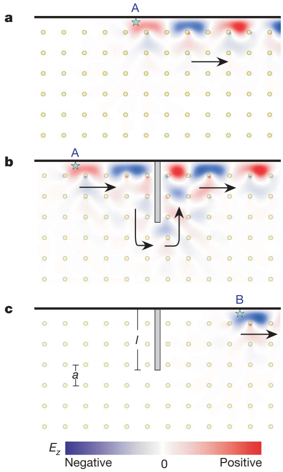

Joannopoulos, Soljačić et al (2009)

- Photonic bulk-edge correspondences

$\downarrow$

- Identify topological observables
  $O = T + \text{error}$

$\downarrow$

- Find all topological invariants T

$\downarrow$

- Classification of PhCs by symmetries

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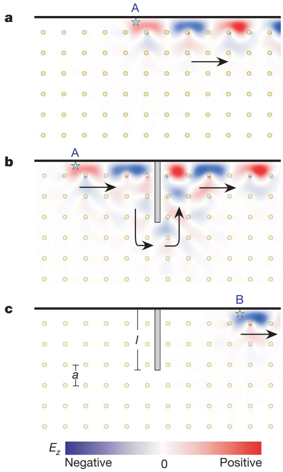

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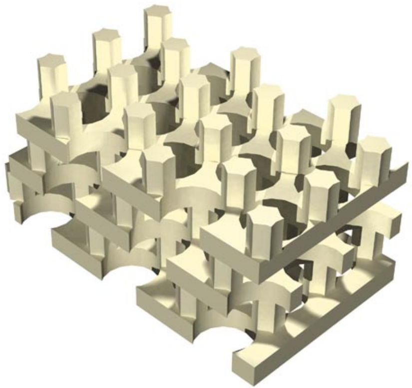
- **Classification of PhCs by symmetries**

Fundamental Equations

Maxwell's Equations in Matter

Maxwell's Equations for Non-Gyrotropic Dielectrics



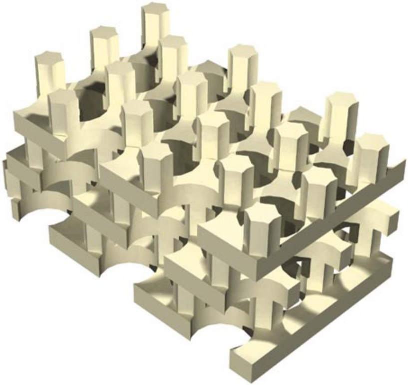
Johnson & Joannopoulos (2004)

Assumption (Material weights)

$$W(x) = \begin{pmatrix} \varepsilon(x) & 0 \\ 0 & \mu(x) \end{pmatrix}$$

1. $W = \overline{W}$ real (non-gyrotropic)
2. $W^* = W$ (lossless)
3. $0 < c\,\mathbf{1} \leq W \leq C\,\mathbf{1}$ (excludes metamaterials)
4. W frequency-independent (response instantaneous)

Maxwell's Equations for Non-Gyrotropic Dielectrics



Johnson & Joannopoulos (2004)

Maxwell equations

Dynamical equations

$$\begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} -\nabla \times \mathbf{H} \\ +\nabla \times \mathbf{E} \end{pmatrix}$$

Absence of sources

$$\begin{pmatrix} \mathrm{div} & 0 \\ 0 & \mathrm{div} \end{pmatrix} \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = 0$$

Schrödinger Formalism of Electromagnetism

$$\begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} -\nabla \times \mathbf{H} \\ +\nabla \times \mathbf{E} \end{pmatrix}$$

dynamical Maxwell equations $\Longleftrightarrow$

$$\mathrm{i}\partial_t \Psi = M \Psi$$

"Schrödinger-type equation"

$$\Psi(t) = \big(\mathbf{E}(t), \mathbf{H}(t)\big) \in L^2_W(\mathbb{R}^3, \mathbb{C}^6)$$ transversal em field

$$M = \underbrace{\begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix}^{-1}}_{= W^{-1}} \underbrace{\begin{pmatrix} 0 & +(-\mathrm{i}\nabla)^{\times} \\ -(-\mathrm{i}\nabla)^{\times} & 0 \end{pmatrix}}_{= \mathrm{Rot}} = M^*$$

Maxwell equations $\Longleftrightarrow$ Maxwell operator $M = M^*$ $\Longrightarrow$ Adaptation of **techniques from quantum mechanics** to electromagnetism

Fundamental Symmetries of Non-Gyrotropic Materials

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3 Symmetries

1. $C : (\mathbf{E}, \mathbf{H}) \mapsto (\overline{\mathbf{E}}, \overline{\mathbf{H}})$ with $C\, M\, C = -M$ (+PH)
2. $J : (\mathbf{E}, \mathbf{H}) \mapsto (\mathbf{E}, -\mathbf{H})$ with $J\, M\, J = -M$ (χ)
3. $T = J\, C : (\mathbf{E}, \mathbf{H}) \mapsto (\overline{\mathbf{E}}, -\overline{\mathbf{H}})$ with $T\, M\, T = +M$ (+TR)

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Restriction to Real Fields

$C\,M\,C = -M$ implies

$$\mathrm{e}^{-\mathrm{i}t\,M}\,(\mathbf{E}_0, \mathbf{H}_0) = \mathrm{e}^{-\mathrm{i}t\,M}\,\mathrm{Re}\,\Psi_{\pm} = \mathrm{Re}\;\mathrm{e}^{-\mathrm{i}t\,M}\Psi_{\pm}$$

where $\mathrm{Re} := \frac{1}{2}(\mathrm{id} + C)$ is the real part operator and

$$\Psi_{+} = 1_{\{\omega>0\}}(M)\,(\mathbf{E}_0, \mathbf{H}_0) = P_{+}(\mathbf{E}_0, \mathbf{H}_0)$$
$$\Psi_{-} = 1_{\{\omega<0\}}(M)\,(\mathbf{E}_0, \mathbf{H}_0) = P_{-}(\mathbf{E}_0, \mathbf{H}_0) = C\Psi_{+}$$

the positive and negative frequency contributions

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What About Gyrotropic Media?

What if

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is complex?

1. Use non-gyrotropic equations $\begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} -\nabla \times \mathbf{H} \\ +\nabla \times \mathbf{E} \end{pmatrix}$
   $\rightsquigarrow$ often implicitly use in literature, but $\mathrm{Im}\,(\mathbf{E}(t), \mathbf{H}(t)) \neq 0$ ↯
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The Schrödinger Formalism for Gyrotropic Media

$\Psi_-(t) = C\Psi_+(t)$ can be enforced by choosing $W_- = \overline{W_+}$, i. e.

$$M_\pm = -C\, M_\mp\, C = W_\pm \, \mathrm{Rot}\big|_{\pm\omega>0}$$

Relation between $M_\pm$ implies relation between evolution groups:

$$C\, \mathrm{e}^{-\mathrm{i} M_\pm} = \mathrm{e}^{-\mathrm{i} t\, M_\mp}\, C$$

The Schrödinger Formalism for Gyrotropic Media

Maxwell equations equivalent to

$$\mathrm{i}\partial_t \Psi(t) = M \Psi(t), \qquad \Psi(0) = \Phi \in \mathcal{H},$$

on the Hilbert space

$$\mathcal{H} := \operatorname{ran} P_+ \oplus \operatorname{ran} P_- \subset L^2_{W_+}(\mathbb{R}^3, \mathbb{C}^6) \oplus L^2_{W_-}(\mathbb{R}^3, \mathbb{C}^6)$$

with Maxwell operator

$$\begin{aligned} M &:= M_+ \oplus M_- \\ \mathcal{D}(M) &:= \big(P_+ \mathcal{D}(\mathrm{Rot})\big) \oplus \big(P_- \mathcal{D}(\mathrm{Rot})\big) \end{aligned}$$

"Indestructible" Symmetries

$$M = M_+ \oplus M_- \implies \begin{cases} K\,M\,K = -M \\ \Gamma\,M\,\Gamma = +M \end{cases}$$

has an **even particle-hole-type symmetry**

$$K := \sigma_1 \otimes C, \quad (\Psi_+, \Psi_-) \mapsto (\overline{\Psi_-}, \overline{\Psi_+}),$$

which *translates to complex conjugation of fields* and a **grading**

$$\Gamma := \sigma_3 \otimes \mathrm{id}, \quad (\Psi_+, \Psi_-) \mapsto (+\Psi_+, -\Psi_-).$$

Reduction to Complex Fields with $\omega > 0$

Physically only **real states** relevant

$$\mathcal{H}_{\mathbb{R}} := \Big\{ (\Psi, \overline{\Psi}) \;\big|\; \Psi \in \mathrm{ran}\, P_+ \Big\} \subset \mathrm{ran}\, P_+ \oplus \mathrm{ran}\, P_-$$

$K \, M \, K = -M$ implies

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Real states $\Longleftrightarrow$ Complex states with $\omega > 0$

$\Longrightarrow$ Study symmetries of M_+ (regular, $\pm$TR)

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Which Symmetries *Are* Broken?

Non-Gyrotropic Materials

$$W_+ = \overline{W_+}$$

1 Relevant Symmetry of Complexified Equation

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$\Longrightarrow$ *Needs to be broken to have unidirectional edge modes!*

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Main Result
Classification of PTIs

Classification for the Periodic Case

Reduction to Real Fields

Topology of the Bloch Bundle

$$\mathcal{E}_{\mathrm{Bloch}} = \mathcal{E}_{+} \oplus \mathcal{E}_{-} \cong \mathcal{E}_{+} \oplus \mathcal{E}_{+}^{*} \longrightarrow \mathbb{T}^{3}$$

Real states determined by component in $\mathcal{E}_{+}$

$\Longrightarrow$ Classification of real states $\rightsquigarrow$ **topology of $\mathbb{C}$-vector bundle** $\mathcal{E}_{+}$

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Comparison Between Photonics and Quantum Mechanics

Theorem (De Nittis-L., 2016)

Material	Photonics	Quantum Mechanics
ordinary	class AI +TR	class AI +TR
exhibiting edge currents	class A none	class A/AII none/-TR
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Thank you for your attention!