Periodic operators with defects of smaller dimensions. Spectral problem

A. A. Kutsenko

Department of Mathematics, Aarhus University, Denmark

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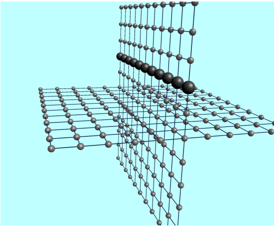
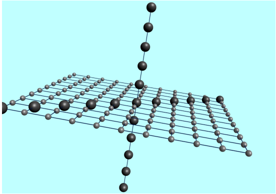
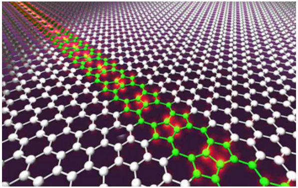
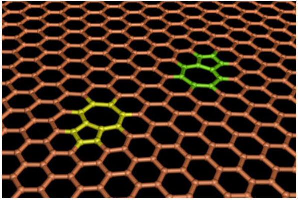
Example of periodic lattices with defects

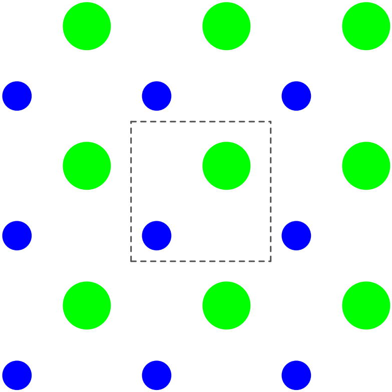








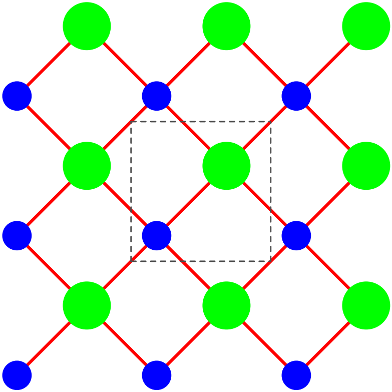
Periodic lattices



We can define N-periodic lattice with M-point unit cell as follows

$$\Gamma = [1, ..., M] \times \mathbb{Z}^N.$$

Periodic operators



Any (bounded) operator

$$\mathcal{A} : \ell^2(\Gamma) \to \ell^2(\Gamma)$$

which commutes with all shift operators

$$\mathcal{S}_{\mathbf{m}} u(j, \mathbf{n}) = u(j, \mathbf{n}+\mathbf{m}), \quad u \in \ell^2(\Gamma)$$

is called a periodic operator.

Fourier-Floquet-Bloch transformation

The corresponding transformation based on Fourier series

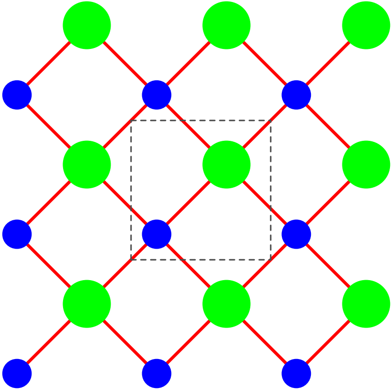
$$\mathcal{F} : \ell^2(\Gamma) \to L^2_{N,M} := L^2([0,1]^N, \mathbb{C}^M),$$

$$(\mathcal{F}u)_j(\mathbf{k}) = \sum_{\mathbf{n}\in\mathbb{Z}^N} e^{2\pi i \mathbf{k}\cdot\mathbf{n}} u(j,\mathbf{n})$$

allows us to rewrite our periodic operator $\mathcal{A}$ as an operator of multiplication by a matrix-valued function $\mathbf{A}$

$$\hat{\mathcal{A}} := \mathcal{F}\mathcal{A}\mathcal{F}^{-1} : L^2_{N,M} \to L^2_{N,M}, \qquad \hat{\mathcal{A}}\mathbf{u} = \mathbf{A}\mathbf{u}.$$

Periodic operators after F-F-B transformation



A periodic operator $\mathcal{A}$ unitarily equivalent to the following operator

$$\hat{\mathcal{A}} : L^2_{N,M} \to L^2_{N,M},$$

$$\hat{\mathcal{A}}\mathbf{u}(\mathbf{k}) = \mathbf{A}_0(\mathbf{k})\mathbf{u}(\mathbf{k})$$

with some (usually continuous) $M \times M$ matrix-valued function $\mathbf{A}_0(\mathbf{k})$ depending on the "quasi-momentum" $\mathbf{k} \in [0,1]^N$.

Spectrum of periodic operators

For the operator of multiplication by the matrix-valued function

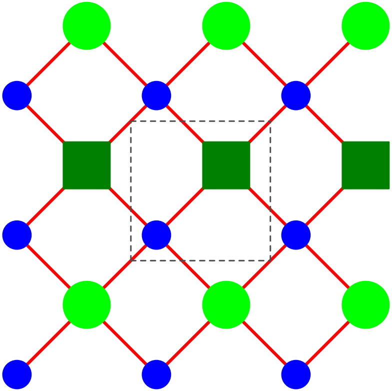
$$\hat{\mathcal{A}}\mathbf{u}(\mathbf{k}) = \mathbf{A}_0(\mathbf{k})\mathbf{u}(\mathbf{k})$$

the spectrum is just eigenvalues of this matrix for different quasi-momentums

$$\sigma(\hat{\mathcal{A}}) = \{\lambda : \ \det(\mathbf{A}_0(\mathbf{k}) - \lambda \mathbf{I}) = 0 \ \textit{for some } \mathbf{k}\} =$$

$$\bigcup_{j=1}^{M} \bigcup_{\mathbf{k}\in[0,1]^N} \{\lambda_j(\mathbf{k})\}.$$

Periodic operators with linear defects ($N = 2$)



In this case our periodic operator

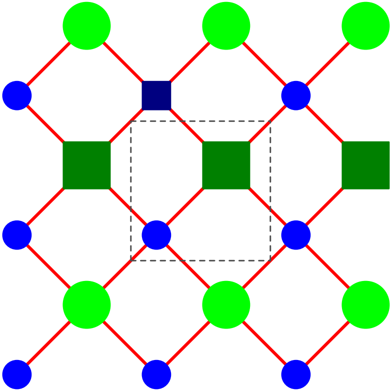
$$\hat{\mathcal{A}} : L^2_{N,M} \to L^2_{N,M},$$

takes the form

$$\hat{\mathcal{A}}\mathbf{u} = \mathbf{A}_0\mathbf{u} + \mathbf{A}_1\langle \mathbf{B}_1\mathbf{u}\rangle_1$$

with some (usually continuous) matrix-valued functions $\mathbf{A}$, $\mathbf{B}$ and

$$\langle \cdot \rangle_1 := \int_0^1 \cdot\, dk_1.$$



In this case our periodic operator

$$\hat{\mathcal{A}} : L^2_{N,M} \to L^2_{N,M},$$

takes the form

$$\hat{\mathcal{A}}\mathbf{u} = \mathbf{A}_0\mathbf{u} + \mathbf{A}_1\langle \mathbf{B}_1\mathbf{u}\rangle_1 + \mathbf{A}_2\langle \mathbf{B}_2\mathbf{u}\rangle_2$$

with some (usually continuous) matrix-valued functions **A**, **B** and

$$\langle \cdot \rangle_2 := \int_0^1 \int_0^1 \cdot \, dk_1 dk_2.$$

Periodic operator with defects (general case)

In general, a periodic operator with defects is unitarily equivalent to the operator $\hat{\mathcal{A}} : L^2_{N,M} \to L^2_{N,M}$ of the form

$$\hat{\mathcal{A}}\mathbf{u} = \mathbf{A}_0\mathbf{u} + \mathbf{A}_1\langle \mathbf{B}_1\mathbf{u}\rangle_1 + ... + \mathbf{A}_N\langle \mathbf{B}_N\mathbf{u}\rangle_N.$$

with continuous matrix-valued functions $\mathbf{A}$, $\mathbf{B}$ and

$$\langle\cdot\rangle_1 = \int_0^1 \cdot dk_1, \quad \langle\cdot\rangle_{j+1} = \int_0^1 \langle\cdot\rangle_j dk_{j+1}.$$

Remark. The spectrum of this operator is

$$\sigma(\mathcal{A}) = \{\lambda : \ \mathcal{A} - \lambda\mathcal{I} \ is\ non-invertible\} = \{\lambda : \ \widetilde{\mathcal{A}} \ is\ non-invertible\},$$

where $\widetilde{\mathcal{A}}$ has the same form as $\mathcal{A}$ but with $\mathbf{A}_0 - \lambda\mathbf{I}$ instead of $\mathbf{A}_0$.

Test for invertibility of a periodic operator with defects

Theorem (J. Math. Anal. Appl., 2015)

Step 0. *Define* $\pi_0 = \det \mathbf{E}_0, \quad \mathbf{E}_0 = \mathbf{A}_0.$

If $\pi_0(\mathbf{k}^0) = 0$ *for some* $\mathbf{k}^0 \in [0,1]^N$ *then* $\mathcal{A}$ *is non-invertible else define* $\mathbf{A}_{j0} = \mathbf{A}_0^{-1}\mathbf{A}_j, \quad j = 1, ..., N.$

Step 1. *Define* $\pi_1 = \det \mathbf{E}_1, \quad \mathbf{E}_1 = \mathbf{I} + \langle \mathbf{B}_1 \mathbf{A}_{10} \rangle_1.$

If $\pi_1(\mathbf{k}_1^0) = 0$ *for some* $\mathbf{k}_1^0 \in [0,1]^{N-1}$ *then* $\mathcal{A}$ *is non-invertible else define* $\mathbf{A}_{j1} = \mathbf{A}_{j0} - \mathbf{A}_{10}\mathbf{E}_1^{-1}\langle \mathbf{B}_1 \mathbf{A}_{j0} \rangle_1, \quad j = 2, ..., N.$

Step 2. *Define* $\pi_2 = \det \mathbf{E}_2, \quad \mathbf{E}_2 = \mathbf{I} + \langle \mathbf{B}_2 \mathbf{A}_{21} \rangle_2.$

If $\pi_2(\mathbf{k}_2^0) = 0$ *for some* $\mathbf{k}_2^0 \in [0,1]^{N-2}$ *then* $\mathcal{A}$ *is non-invertible else define* $\mathbf{A}_{j2} = \mathbf{A}_{j1} - \mathbf{A}_{21}\mathbf{E}_2^{-1}\langle \mathbf{B}_2 \mathbf{A}_{j1} \rangle_2, \quad j = 3, ..., N.$

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Step N. *Define* $\pi_N = \det \mathbf{E}_N, \quad \mathbf{E}_N = \mathbf{I} + \langle \mathbf{B}_N \mathbf{A}_{N,N-1} \rangle_N.$ *If* $\pi_N = 0$ *then* $\mathcal{A}$ *is non-invertible else* $\mathcal{A}$ *is invertible.*

Determinants in the case of embedded defects

In this case the operator has a form

$$\mathcal{A}\cdot = \mathbf{A}_0 \cdot + \mathbf{A}_1 \langle \cdot \rangle_1 + ... + \mathbf{A}_N \langle \cdot \rangle_N,$$

where $\mathbf{A}_n$ does not depend on $k_1, ..., k_n$.
Define the matrix-valued integral continued fractions

$$\mathbf{C}_0 = \mathbf{A}_0, \quad \mathbf{C}_1 = \mathbf{A}_1 + \left\langle \frac{\mathbf{I}}{\mathbf{A}_0} \right\rangle_1^{-1}, \quad \mathbf{C}_2 = \mathbf{A}_2 + \left\langle \frac{\mathbf{I}}{\mathbf{A}_1 + \left\langle \frac{\mathbf{I}}{\mathbf{A}_0} \right\rangle_1^{-1}} \right\rangle_2^{-1}$$

and so on $\mathbf{C}_j = \mathbf{A}_j + \langle \mathbf{C}_{j-1}^{-1} \rangle_j^{-1}$. Then

$$\pi_j(\mathcal{A}) = \det(\langle \mathbf{C}_{j-1}^{-1} \rangle_j \mathbf{C}_j).$$

Note that if all $\mathbf{A}_j$ are self-adjoint then $\mathcal{A}$ is self-adjoint and all $\mathbf{C}_j$ are self-adjoint. arxiv.org, 2015

Spectrum of periodic operators with defects

The spectrum of $\mathcal{A}$ has the form

$$\sigma(\mathcal{A}) = \bigcup_{n=0}^{N} \sigma_n, \quad \sigma_n = \{\lambda : \ \widetilde{\pi}_n = 0 \ \textit{for some } \mathbf{k}\},$$

$$\textit{where} \quad \widetilde{\pi}_n \equiv \pi_n(\mathcal{A} - \lambda\mathcal{I}) \equiv \pi_n(\lambda, k_{n+1}, ..., k_N).$$

The component σ_0 coincides with the spectrum of purely periodic operator $\mathbf{A}_0\mathbf{u}$ without defects. All components σ_n, $n < N$ are continuous, the component σ_N is discrete.
Also note that σ_n does not depend on the defects of dimensions greater than n, i.e. of $\mathbf{A}_{n+1}$, $\mathbf{B}_{n+1}$, $\mathbf{A}_{n+2}$, $\mathbf{B}_{n+2}$ and so on.

Determinants of periodic operators with defects

For all continuous matrix-valued functions $\mathbf{A}$, $\mathbf{B}$ on $[0,1]^N$ of appropriate sizes introduce

$$\mathfrak{H} = \{\mathcal{A} : \ \mathcal{A} = \mathbf{A}_0 \cdot + \mathbf{A}_1 \langle \mathbf{B}_1 \cdot \rangle_1 + ... + \mathbf{A}_N \langle \mathbf{B}_N \cdot \rangle_N\} \subset \mathcal{B}(L^2_{N,M}),$$

$$\mathfrak{G} = \{\mathcal{A} \in \mathfrak{H} : \ \mathcal{A} \textit{ is invertible}\}.$$

Theorem (arxiv.org, 2015)

The set $\mathfrak{H}$ is a non-closed operator algebra. The subset $\mathfrak{G}$ is a group. The mapping

$$\boldsymbol{\pi}(\mathcal{A}) := (\pi_0(\mathcal{A}), ..., \pi_N(\mathcal{A}))$$

is a group homomorphism between $\mathfrak{G}$ and $\mathcal{C}_0 \times \mathcal{C}_1 \times ... \times \mathcal{C}_N$, where $\mathcal{C}_n$ is a group of non-zero continuous functions depending on $(k_{n+1}, ..., k_N) \in [0,1]^{N-n}$.

Define

$$\boldsymbol{\tau}(\mathcal{A}) = \lim_{t\to 0} \frac{\boldsymbol{\pi}(\mathcal{I} + t\mathcal{A}) - \boldsymbol{\pi}(\mathcal{I})}{t}.$$

Then

Theorem (arxiv.org, 2015)

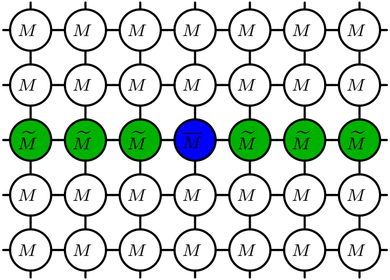
The following identities are fulfilled

$$\boldsymbol{\tau}(\mathcal{A}) = (\operatorname{Tr} \mathbf{A}_0, \langle \operatorname{Tr} \mathbf{B}_1 \mathbf{A}_1 \rangle_1, ..., \langle \operatorname{Tr} \mathbf{B}_N \mathbf{A}_N \rangle_N),$$

$$\boldsymbol{\tau}(\alpha\mathcal{A} + \beta\mathcal{B}) = \alpha\boldsymbol{\tau}(\mathcal{A}) + \beta\boldsymbol{\tau}(\mathcal{B}), \quad \boldsymbol{\tau}(\mathcal{A} \circ \mathcal{B}) = \boldsymbol{\tau}(\mathcal{B} \circ \mathcal{A}),$$

$$\boldsymbol{\pi}(e^{\mathcal{A}}) = e^{\boldsymbol{\tau}(\mathcal{A})}, \quad \boldsymbol{\pi}(\mathcal{A} \circ \mathcal{B}) = \boldsymbol{\pi}(\mathcal{A})\boldsymbol{\pi}(\mathcal{B}).$$

Example. Uniform lattice with guide and single defect.



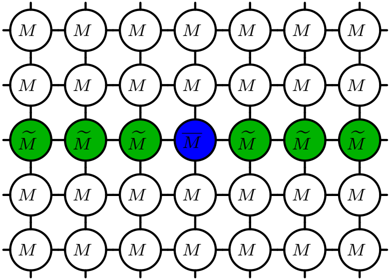
Wave equation has the form ($\lambda = \omega^2$ is an energy)

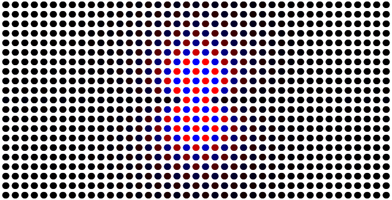
$$-(\Delta_{\text{disc}})u(x,y) = \lambda \begin{cases} \overline{M}u(x,y), & x = y = 0, \\ \widetilde{M}u(x,y), & x = 0,\ y \neq 0, \\ Mu(x,y), & \textit{otherwise} \end{cases} \qquad u \in \ell^2(\mathbb{Z}^2)$$

After applying Floquet-Bloch transformation it becomes

$$4(\sin^2 \pi k_1 + \sin^2 \pi k_2)\hat{u} = \lambda M\hat{u} + \lambda(\widetilde{M} - M)\int_0^1 \hat{u}\, dk_1 + \lambda(\overline{M} - \widetilde{M})\int_0^1\int_0^1 \hat{u}\, dk_1 dk_2$$

Example. Uniform lattice with guide and single defect.





For the random uniform distribution of the masses of the media, of the guide, and of the point defect ($< M$) the probability of existence of the isolated eigenvalue is exactly

$$\frac{3}{4} - \frac{1}{2\pi}.$$

Comput. Mech., 2014

Wave propagation in the lattice with defects and sources

Wave equation

$$\Delta_{\mathrm{discr}} U_{\mathbf{n}}(t) = S_{\mathbf{n}}^2 \ddot{U}_{\mathbf{n}}(t) + \sum_{\mathbf{n}' \in \mathcal{N}_{\mathrm{F}}} F_{\mathbf{n}'}(t) \delta_{\mathbf{n}\mathbf{n}'}, \quad \mathbf{n} \in \mathbb{Z}^2$$

Assuming harmonic sources and applying F-B transformation we obtain

$$Av = -\omega^2 \mathbf{a}^* \mathbf{S} \langle v \mathbf{a} \rangle + \mathbf{b}^* \mathbf{f}.$$

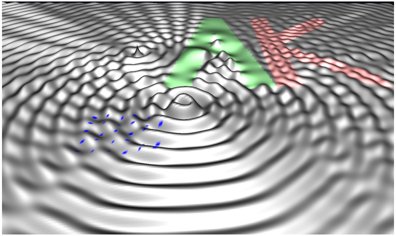
Using determinants we may derive explicit solution of the last equation

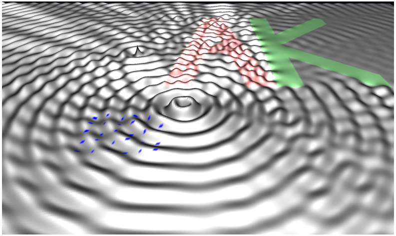
$$v = A^{-1} \left(-\omega^2 \mathbf{a}^* \mathbf{S} \mathbf{G} \left\langle \frac{\mathbf{a}\mathbf{b}^*}{A} \right\rangle + \mathbf{b}^* \right) \mathbf{f},$$

where

$$\mathbf{G} = (\mathbf{I} + \omega^2 \mathbf{A} \mathbf{S})^{-1}, \quad \mathbf{A} = \left\langle \frac{\mathbf{a}\mathbf{a}^*}{A} \right\rangle.$$

Wave simulations. Inverse problem.





Two formulas allows us to recover the defect properties from the information about amplitudes of waves at the receivers

$$\mathbf{S}\langle u\mathbf{a}\rangle = \omega^{-2}\mathbf{C}^{-1}\left(\left\langle \frac{\mathbf{c}\mathbf{b}^*}{A}\right\rangle \mathbf{f} - \langle u\mathbf{c}\rangle\right),$$

$$\langle u\mathbf{a}\rangle = -\mathbf{A}\mathbf{C}^{-1}\left(\left\langle \frac{\mathbf{c}\mathbf{b}^*}{A}\right\rangle \mathbf{f} - \langle u\mathbf{c}\rangle\right) + \left\langle \frac{\mathbf{a}\mathbf{b}^*}{A}\right\rangle \mathbf{f},$$

where

$$\mathbf{c} = \begin{pmatrix} e^{-i\mathbf{n}_1\cdot\mathbf{k}} \\ \dots \\ e^{-i\mathbf{n}_N\cdot\mathbf{k}} \end{pmatrix}_{\mathbf{n}_j\in\mathcal{N}_{\mathrm{R}}}, \quad \mathbf{C} = \left\langle \frac{\mathbf{c}\mathbf{a}^*}{A}\right\rangle.$$

Eur. J. Mech. A-Solid., 2015

Cloaking device.