Bogolubov Theory and the Critical Temperature in the Dilute Limit

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The Canonical Functional

$$\begin{aligned}\mathcal{F}(\gamma,\alpha,\rho_0) &= (2\pi)^{-3}\int_{\mathbb{R}^3} p^2\gamma(p)dp - TS(\gamma,\alpha) + \frac{1}{2}\widehat{V}(0)\rho^2 \\ &+ (2\pi)^{-3}\rho_0\int_{\mathbb{R}^3}\widehat{V}(p)\left(\gamma(p)+\alpha(p)\right)dp \\ &+ (2\pi)^{-6}\frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3}\widehat{V}(p-q)\left(\alpha(p)\alpha(q)+\gamma(p)\gamma(q)\right)dpdq,\end{aligned}$$

Here V radial, sufficiently fast decaying, sufficiently smooth, and $V, \widehat{V} \geq 0$.
The variational problem is

$$\inf\{\mathcal{F}(\gamma,\alpha,\rho_0) \mid \rho_0 \geq 0, \gamma \geq 0, \alpha^2 \leq \gamma(1+\gamma)\}$$

General Results

Theorem (Existence of canonical minimizers)

Given $\rho > 0$ and $T \geq 0$. Under the assumptions on V the variational problem admits a minimizer (γ, α, ρ_0) (which is not necessarily unique).

Theorem (Equivalence of BEC and superfluidity)

Let (γ, α, ρ_0) be a minimizing triple. Then

$$\rho_0 = 0 \iff \alpha \equiv 0.$$

Theorem (Existence of canonical phase transition)

For fixed $\rho > 0$ there exist temperatures $0 < T_1 < T_2$ such that any minimizing triple (γ, α, ρ_0)

1. $\rho_0 = 0$ *if* $T \geq T_2$;
2. $\rho_0 > 0$ *if* $0 \leq T \leq T_1$.

The Free Gas

$$\mathcal{F}_0(\gamma) = (2\pi)^{-3} \int p^2\gamma(p) - Ts(\gamma(p), 0)dp.$$

Minimizer

$$\gamma_\mu(p) = \frac{1}{e^{(p^2-\mu)/T} - 1},$$

$$\rho = (2\pi)^{-3}T^{3/2} \int \frac{1}{e^{(p^2 - T^{-1}\mu(\rho))} - 1} dp \leq (8\pi^{3/2})^{-1}\zeta(3/2)T^{3/2} = \rho_{\rm fc},$$

Minmizing free energy

$$F_0(T,\rho) = (2\pi)^{-3}T \int \ln\left(1 - e^{-(p^2-\mu(\rho))/T}\right) dp + \mu(\rho)\rho$$

$$F_0(T,\rho) = T^{5/2} f_0\left(\rho/T^{3/2}\right), \qquad \mu(\rho) = Tm\left(\rho/T^{3/2}\right),$$

The phase transition is 3rd order:

$$f_0(n) = -\left(8\pi^{3/2}\right)^{-1}\zeta(5/2) + C_1[n_{\rm fc} - n]_+^3 + o([n_{\rm fc} - n]_+^3)$$

The Dilute Limit

The dilute limit $\rho^{1/3}a \to 0$. The scattering length:

$$4\pi a := \int \Delta w = \frac{1}{2}\int Vw,$$

Scattering solution w determined by

$$-\Delta w + \frac{1}{2}Vw = 0, \quad w(x) \to 1 \text{ as } |x| \to \infty$$

Apriori estimate

$$\gamma \sim \gamma_{\text{free}}(p) \sim (e^{p^2/T} - 1)^{-1}$$

Hence

$$\sqrt{T} \sim \left(\int \gamma(p)dp\right)^{1/3} \sim \rho^{1/3} \ll a^{-1}$$

Approximation Errors

$$E_1 = \left|(2\pi)^{-3}\rho_0 \int \gamma(p)\widehat{V}(p)dp - \widehat{V}(0)\rho_0\rho_\gamma\right|$$

$$E_2 = \left|(2\pi)^{-3}\rho_0 \int \gamma(p)\widehat{Vw}(p)dp - \widehat{Vw}(0)\rho_0\rho_\gamma\right|$$

$$E_3 = \Big|(2\pi)^{-6}\frac{1}{2}\iint \gamma(p)\widehat{V}(p-q)\gamma(q)dpdq - \frac{1}{2}\widehat{V}(0)\rho_\gamma^2 - \frac{\zeta(3/2)\zeta(5/2)}{256\pi^3}\Delta\widehat{V}(0)T^4\Big|.$$

$$E_4 = (2\pi)^{-6}\frac{1}{2}\iint (\alpha-\alpha_0)(p)\widehat{V}(p-q)(\alpha-\alpha_0)(q)dpdq$$

$$\alpha_0 = (\rho_0+t_0)\widehat{w} - (2\pi)^3\rho_0\delta_0 = (2\pi)^3 t_0\delta_0 - \frac{\rho_0+t_0}{2}\frac{\widehat{Vw}(p)}{p^2},$$

α_0 approximation to α, which is actually more complicated near $p=0$ but in the double integral $t_0\delta_0$ suffices.

Approximate Functional

We find the approximate functional

$$\mathcal{F}^{\rm app}(\gamma,\alpha,\rho_0) = (2\pi)^{-3}\int \left(p^2+(\rho_0+t_0)\widehat{Vw}(p)\right)\gamma(p)dp$$
$$+(2\pi)^{-3}\int(\rho_0+t_0)\widehat{Vw}(p)\alpha(p)dp - TS(\gamma,\alpha)$$
$$+\frac{1}{4}(2\pi)^{-3}(\rho_0+t_0)^2\int\frac{\widehat{Vw}(p)^2}{p^2}dp+\widehat{V}(0)\rho^2$$
$$+(12\pi a-\widehat{V}(0))\rho_0^2-8\pi a\rho\rho_0-4\pi a t_0^2-8\pi a t_0(\rho-\rho_0)$$
$$+\frac{\zeta(3/2)\zeta(5/2)}{256\pi^3}\Delta\widehat{V}(0)T^4.$$

Minimize the approximate functional

We minimize the approximate functional for fixed ρ_0 and find

$$\inf \mathcal{F} = -(8\pi^{3/2})^{-1}\zeta(5/2)T^{5/2} + \widehat{V}(0)\rho^2 + T^4a^3\left[\frac{1}{8\pi}\left(\frac{(\sigma-k)^3}{12} - \sigma^2\Big(\frac{1}{2} + \frac{1}{2+\sigma-k}\Big)\right) - (\nu - 8\pi)\frac{\sigma^2}{(8\pi)^2}\right] + o(T^4a^3),$$

where we ignored the last constant in the approximate functional, which is of order T^4a^3. Here $\nu = \widehat{V}(0)/a > 8\pi$ and we have introduced dimensionless parameters

$$\rho = \rho_{\rm fc} + \frac{k}{8\pi}T^2a, \qquad \rho_0 = \frac{\sigma}{8\pi}T^2a$$

and t_0 was determined by the consistency equation

$$t_0 = \frac{2\sigma}{8\pi(k-\sigma-2)}T^2a$$

Minimizing over ρ_0

We minimize over ρ_0, i.e., σ

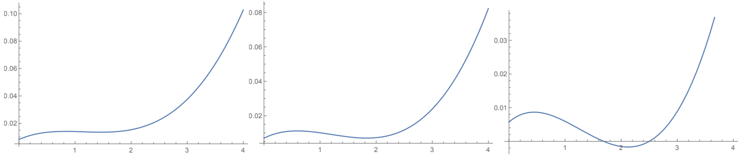

Figure: The minimization in σ. $\nu = 8\pi$, $k = -1.35$, $k = -1.28$ and $k = -1.2$ (from left to right). For $k = -1.28$ there are two minimizers: $\sigma = 0$ and $\sigma = 1.83$.

$$T_{\rm c} = T_{\rm fc}\left(1 + 1.49(\rho^{1/3}a) + o(\rho_{\rm fc}^{1/3}a)\right).$$

Monte Carlo for the full many-body problem gives the same with coefficient 1.3.

The Free Energy Curve

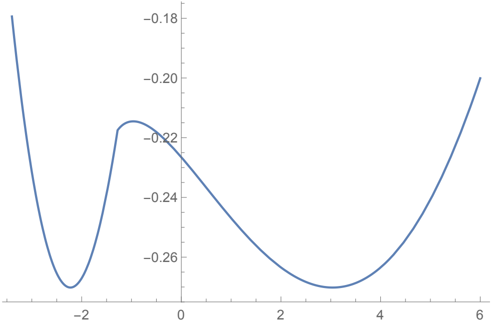

Figure: The curve shows the energy$/a^3T^4$ as a function of k (ρ) tilted. The derivative has a discontinuity at $k = -1.28$, which corresponds to the value of k where σ becomes non-zero.

Literature references for all lectures

The Lecutres have covered part of the material in the following papers:

1. E.H. Lieb and J.P. Solovej, Ground State Energy of the One-Component Charged Bose Gas, Commun. Math. Phys. **217** Issue 1 (2001) pp 127–163 (Erratum: Commun. Math. Phys. **225**, pp 219–221), (2002) [Archive: cond-mat/0007425]
2. E.H. Lieb and J.P. Solovej, Ground State Energy of the Two-Component Charged Bose Gas. Commun. Math. Phys. **252**, 485 – 534, (2004) [Archive: math-ph/0311010]
3. J.P. Solovej, Upper Bounds to the Ground State Energies of the One- and Two-Component Charged Bose Gases, Commun. Math. Phys. **266**, Number 3, 797–818, (2006) [Archive: math-ph/0406014]
4. M. Lewin, P.T. Nam, S. Serfaty, and J.P. Solovej, Bogoliubov spectrum of interacting Bose gases. Comm. Pure and Applied Math., **68**, 3, 413–471, (2015), doi: 10.1002/cpa.21519 [Archive: arXiv:1211.2778]
5. M. Napiórkowski, P. T. Nam, and J.P. Solovej, Diagonalization of bosonic quadratic Hamiltonians by Bogoliubov transformations. To appear in Jour. Func. Analysis. [Archive: arXiv:1508.07321]
6. M. Napiórkowski, R. Reuvers, and J.P. Solovej, The Bogoliubov free energy functional I. Existence of minimizers and phase diagram. [Archive: arXiv:1511.05935]
7. M. Napiórkowski, R. Reuvers, and J.P. Solovej, The Bogoliubov free energy functional II. The dilute limit. [Archive: arXiv:1511.05953]

Thanks

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