University of Trento

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School on «Mathematical Challenges in Quantum Mechanics»

CNR-INO

GROSS-PITAEVSKII EQUATION FOR COUPLED BOSE-EINSTEIN CONDENSATES: SOLITONIC SOLUTIONS

Sandro Stringari

Gross-Pitaevskii equation:

(**Exact** theory for T=0 weakly interacting Bose gases)

Follows from Schrodinger equation,

(Lieb,Seiringer, Yingvanson, 2000; Erdos, Schlein and Yau, 2010)

$$\delta E = 0$$

$$E = \int d\vec{r} \left[\frac{\hbar^2}{2m} |\nabla\Psi|^2 + \frac{1}{2} g |\Psi|^4 - \mu |\Psi|^2 \right]$$

$$g = \frac{4\pi\hbar^2 a}{m}$$ ← **s-wave two-body scattering length**

- Easy generalization to the time dependent case

$\Psi \equiv$ order parameter $= \langle \hat{\Psi} \rangle$ ($\hat{\Psi}$ is quantum field operator)

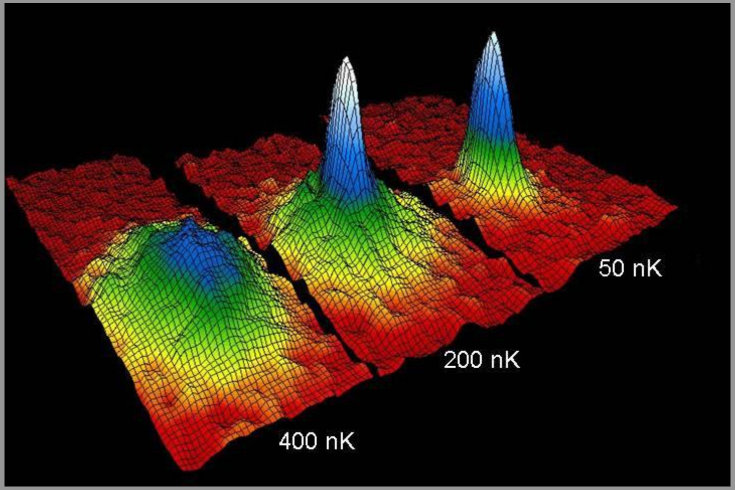
(not to be confused with many body wavefunction).

Follows from spontaneous breaking of gauge symmetry

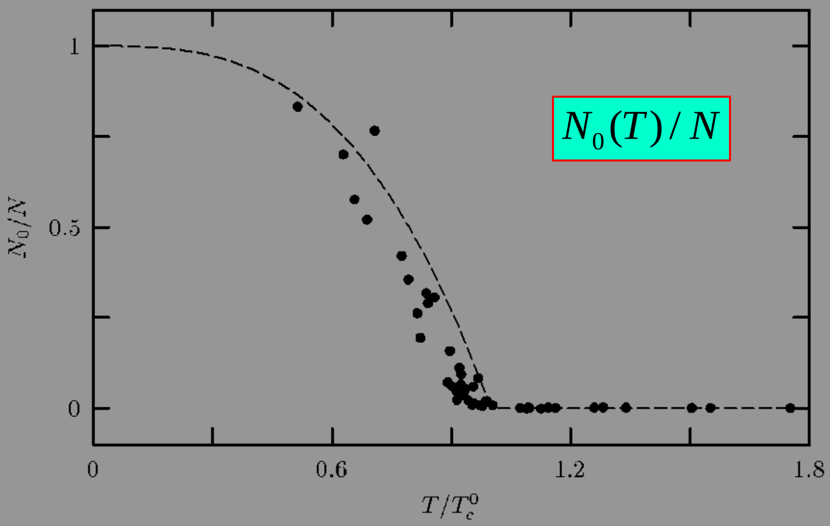
$n_0 \equiv |\Psi|^2$ Bose - Einstein condensate density

(concides with total density at $T = 0$ in 3D weakly interacting gases)

Experimental observation of Bose-Einstein condensation In weakly interacting atomic gases

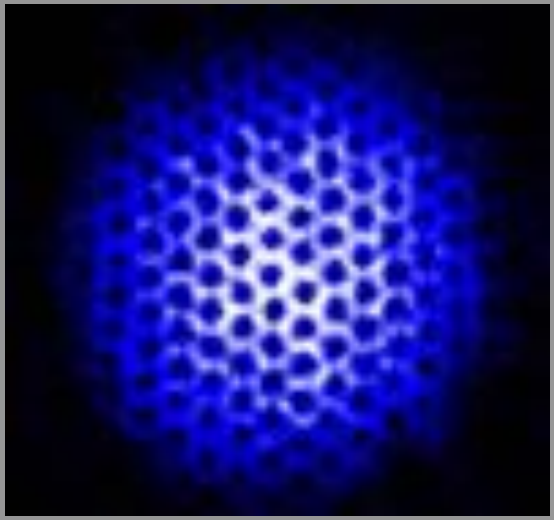


Cornell, Weiman, Ketterle, 1995
Nobel Prize in Physics, 2001



Evidence for phase transition
(Jila 1996)

Some relevant **experimental** features accounted for from Gross-Pitaevskii theory



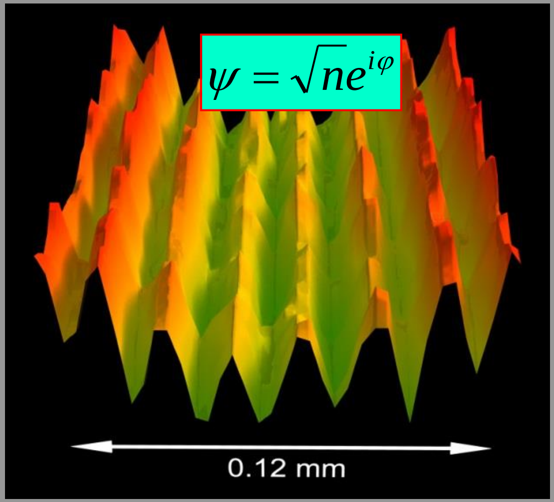
Lattice of **vortices** following from the stirring of the condensate

(Jila, Mit, 2002)

Role of **interactions** and

non-linearity in GP eq.

(separation between vortex lines fixed by quantum of circulation h/m



Interference fringes in

overlapping BECs after expansion

(Mit, 1997)

Role of the **phase** of order parameter

(wave length $\lambda = ht/md$

depends on Planck constant)

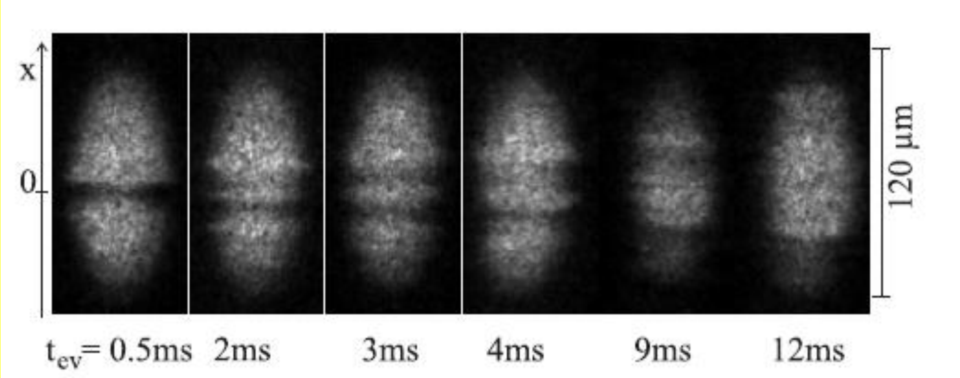
Solitonic solutions of

Gross-Pitaevskii equations

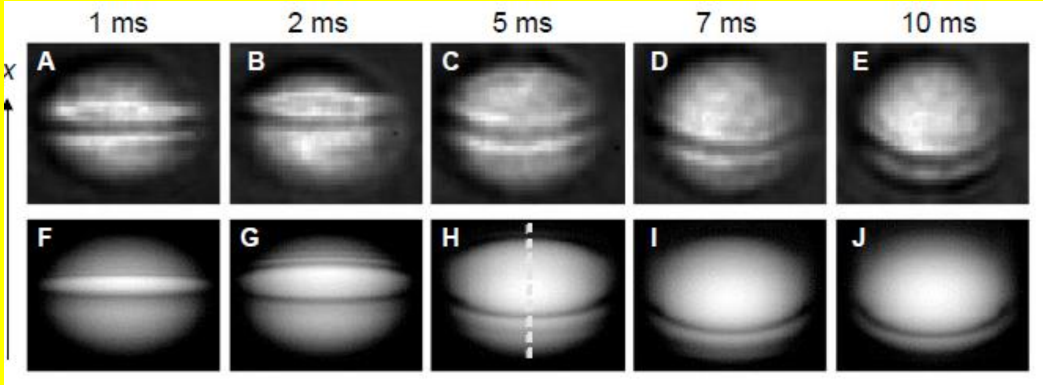
(role of **non-linearity** and **interactions**)

- Solitons are **ubiquitous** non-linear phenomena characterizing different branches of science as diverse as mathematical physics, particle physics, molecular biology, geology, eanography, astrophysics, nonlinear optics etc..
- In ultracold atomic gases solitons can be engineered by **phase imprinting**, **density** imprinting, quantum **quenches**.
- They were measured soon after the realization of BEC.

- Burger et al. 1999



- Denschlag et al. 2000



Summary of the talk:

- Brief introduction to solitons in single Bose-Einstein condensates

- Solitons in two interacting Bose-Einstein condensates

- Role of Rabi coupling, domain wall and precession of vortex molecules

- Conclusions

Solitons in single Bose-Einstein condensates

(Summary)

- Few analytical results for solitons available in quantum gases

- **Dark soliton** in Bose-Einstein condensates.
  Analytic solution of Gross-Pitaevskii equation (Tsuzuki, 1971)

$$\Psi(z-vt)=\sqrt{n}\left(i\frac{v}{c}+\sqrt{1-\frac{v^2}{c^2}}\tanh\left[\frac{z-vt}{\sqrt{2}\xi}\sqrt{1-\frac{v^2}{c^2}}\right]\right)$$

- Maximum soliton velocity v fixed by sound velocity c.

- Width of the soliton fixed by healing length $\xi=\sqrt{\frac{\hbar^2}{2mgn}}=\frac{1}{\sqrt{2}}\frac{\hbar}{mc}$

$$\Delta_{soliton}=\frac{\xi}{\sqrt{1-v^2/c^2}}$$

  Width becomes larger and larger as v approaches the sound velocity

- Energy of moving soliton: $\varepsilon=\frac{4}{3}\hbar cn\left(1-\frac{v^2}{c^2}\right)^{3/2}$ corresponds to **negative effective mass** responsible for **snake instability**. Solitons are stable only in tight radial traps.

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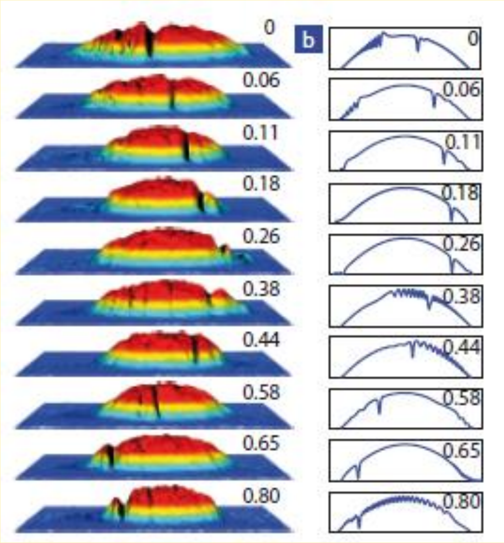
- A dark soliton in a 1D harmonic trap oscillates with frequency

$$\omega = \omega_z / \sqrt{2}$$

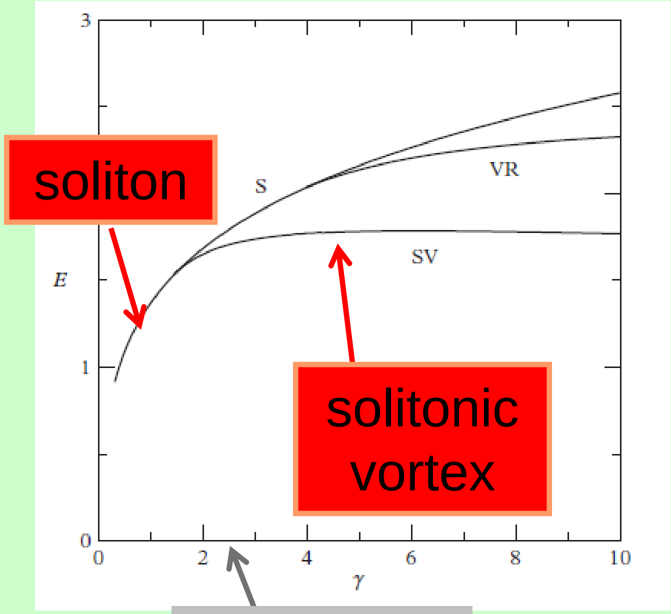
Busch and Anglin (2001)
Konotop and Pitaevskii (2004)

Frequency of oscillation is **independent** of amplitude of oscillation



Measurement of oscillating dark soliton
Becker et al. (2008)

If radial trapping is not tight enough the soliton exhibits snake instability and the new stable topological configuration is a **solitonic vortex**



$$an_1 \propto \left(\frac{R_\perp}{a}\right)^2$$

Komineas and Papanicolau, 2003

Exp identification of solitonic vortex Donadello et al, 2014

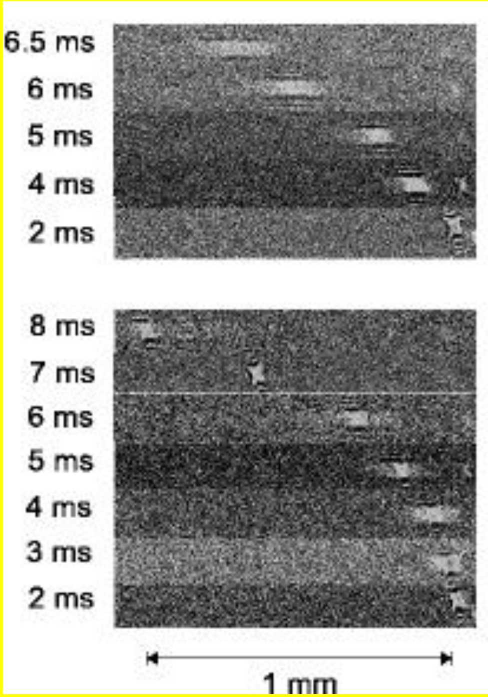
Bright solitons

In addition to dark solitons GP equation admits bright soliton solutions for negative values of the scattering length:

$$\Psi(z) = \Psi(z=0)\frac{1}{\cosh[z/\sqrt{2}\xi]}$$

$$\xi = \sqrt{\frac{\hbar^2}{2m|g|n}}$$

Bright solitons were measured by Khaykovich et al (2002)



non interacting BEC

bright soliton

Solitons in two coupled Bose-Einstein condensates

Realization of multicomponent atomic mixtures is opening new perspectives in the study of solitons in quantum gases

$$i\hbar\partial_t \Psi_1 = \left(H_0 + g_{11} |\Psi_1|^2 + g_{12} |\Psi_2|^2\right)\Psi_1$$

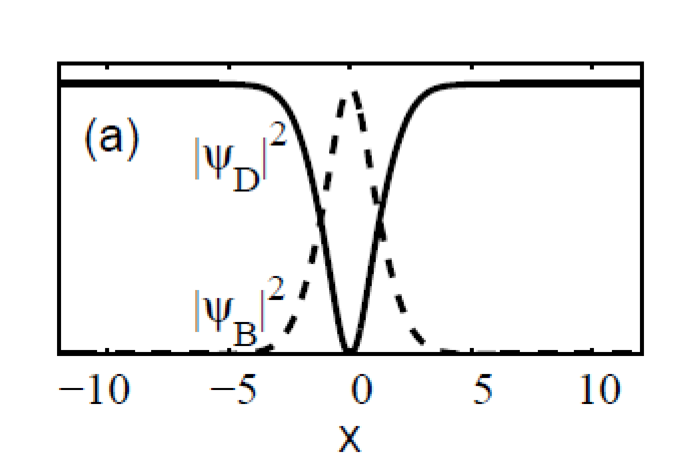
$$i\hbar\partial_t \Psi_2 = \left(H_0 + g_{22} |\Psi_2|^2 + g_{12} |\Psi_1|^2\right)\Psi_2$$

$$H_0 = -\frac{\hbar^2}{2m}\Delta + V_{ext}$$

$\frac{g_{11}g_{22}}{g_{12}^2} \geq 1$ miscibility

$\frac{g_{11}g_{22}}{g_{12}^2} \leq 1$ phase separation

- Analytic solutions for **bright-dark** solitons in two component interacting BEC's were found by Busch and Anglin (2001) assuming equal values for the intraspecies and interspecies coupling constants ($g_{12} = g_{11} = g_{22}$):



$$\Psi_B = \sqrt{\frac{N_B \kappa}{2}} e^{i\phi} e^{i\Omega_B t} e^{ix\kappa \tan\alpha} \operatorname{sec} h[\kappa(x - q(t))]$$

$$\Psi_D = i\sqrt{\mu} \sin\alpha + \sqrt{\mu} \cos\alpha \tanh[\kappa(x - q(t))]$$

where $N_B = \int dz \,|\Psi_B(z)|^2$ is the number of particles per unit length in state B.

$\kappa = \sqrt{\mu \cos^2\alpha + (N_B/4)^2} - N_B/4$ is soliton inverse length

The soliton moves according the law $q(t) = q(0) + t\kappa \tan\alpha$

Magnetic solitons in two coupled Bose-Einstein condensates. Role of spin sound velocity

(Chunley Qu, Lev Pitaevskii and S.S:, in preparation)

We have derived **analytical solutions** for **magnetic solitons** for unequal coupling constants ($g_{11} = g_{22} \equiv g \neq g_{12}$), emphasizing the role of the **spin sound velocity**.

We use the ansatz for the spinor wave function

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \sqrt{n} \begin{pmatrix} \cos(\theta/2)e^{i\varphi_1} \\ \sin(\theta/2)e^{i\varphi_2} \end{pmatrix}.$$

with the total density assumed to be uniform [justified if $\delta g = g - g_{12} << g$ and $\delta g > 0$ (condition of miscibility)].

It is convenient to introduce the relative phase $\varphi_A = \varphi_1 - \varphi_2$ and the total phase $\varphi_A = \varphi_1 + \varphi_2$ of the two order parameters. .

The **Lagrangian** describing the dynamics of a travelling soliton can then be written in the form

$$\tilde{\mathcal{L}} = U\cos\theta\partial_\zeta\varphi_A - \frac{1}{2}[(\partial_\zeta\theta)^2 + \sin^2\theta(\partial_\zeta\varphi_A)^2] + \frac{1}{2}\sin^2\theta$$
$$+U\partial_\zeta\varphi_B - \frac{1}{2}(\partial_\zeta\varphi_B)^2 - \cos\theta(\partial_\zeta\varphi_A)(\partial_\zeta\varphi_B) \qquad (4)$$

with

$$\zeta = (z - Vt)/\xi_S$$

$$\xi_S = \sqrt{\hbar^2/(2mn\delta g)}$$

Spin healing length

Differential equations for $\varphi_A, \varphi_B, \theta$ take the form

$$\partial_\zeta \varphi_B = -\cos\theta \partial_\zeta \varphi_A$$

$$\boxed{\partial_\zeta^2 \vartheta = -\sin\vartheta \cos\theta + U^2 \frac{\cos\theta}{\sin^3\theta}}$$

$$\partial_\zeta \varphi_A = U \frac{\cos\theta}{\sin^2\theta}$$

Eq. for $\theta(\zeta)$ is easily integrated, yielding analytic result for the two density profiles (corresponding to traveling magnetic soliton).

$$\boxed{\begin{aligned} n_1 &= n\cos^2(\theta/2) = \frac{n}{2}\left[1 + \frac{\sqrt{1-U^2}}{\cosh(\zeta\sqrt{1-U^2})}\right] \\ n_2 &= n\sin^2(\theta/2) = \frac{n}{2}\left[1 - \frac{\sqrt{1-U^2}}{\cosh(\zeta\sqrt{1-U^2})}\right] \end{aligned}}$$

$$\zeta = (z - Vt)/\xi_S$$

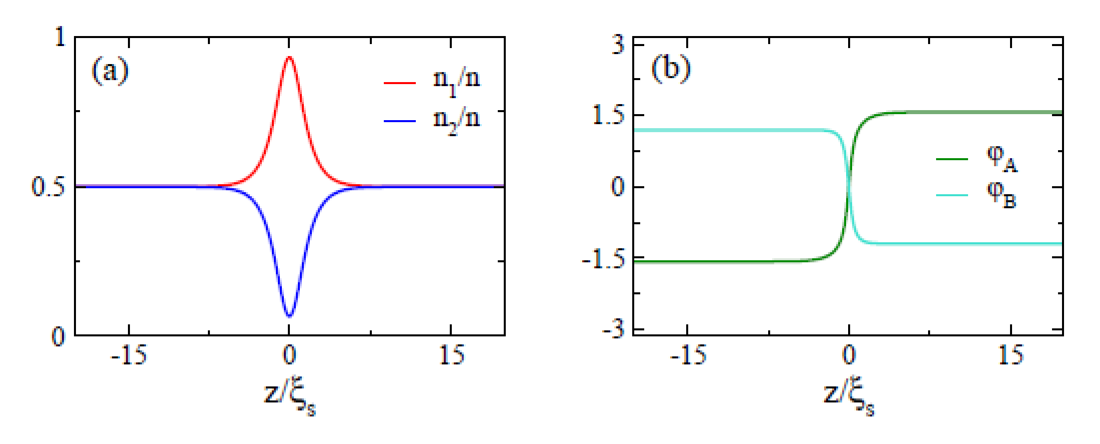
$$U = V/u_S$$

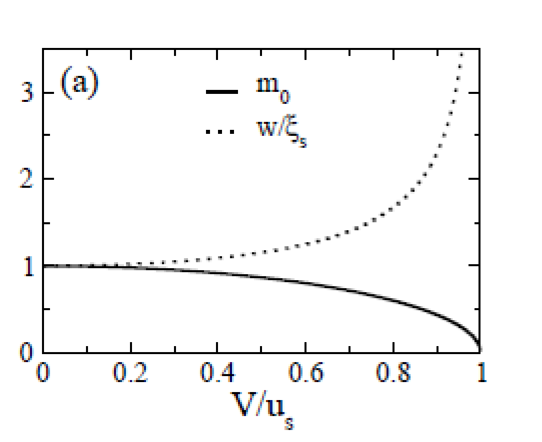
$$u_S = \sqrt{n\delta g/2m}$$

- **Maximum velocity** is **spin sound** velocity
- **Width** of soliton fixed by **spin healing length** and velocity U
- Relative **phase A** has asymptotic jump $\varphi_A(+\infty) - \varphi_A(-\infty) = \pi$ **independent of velocity**
- Asymptotic **jump** $\varphi_B(+\infty) - \varphi_B(-\infty)$ of **total phase B** instead **depends on soliton velocity**

Density profiles, width and relative phase of a magnetic soliton

(Chunley Qu, Lev Pitaevskii and S.S:, in preparation)





- Total density is unperturbed by magnetic soliton
- Magnetization decreases with velocity
- Width increases with velocity
- Jump of relative phase $\varphi_A(+\infty) - \varphi_A(-\infty) = \pi$

  is independent of soliton velocity

Energy of moving soliton depends on velocity according to

$$\varepsilon_S = n\hbar u_S \sqrt{1 - \frac{V^2}{u_S^2}}$$

At small velocity magnetic soliton behaves like a quasiparticle with **negative effective mass**

$$m^* = -\frac{n\hbar}{u_s}$$

Magnetic soliton exhibits snake instability (follows from $m^* < 0$)

(like ordinary dark solitons in single component BECs)

However width of magnetic soliton is **much larger than in ordinary solitions** being fixed by **spin healing length**

$$\boxed{\xi_S = \sqrt{\hbar^2/(2mn\delta g)} >> \sqrt{\hbar^2/(2mng)}}$$

Snake instability is strongly reduced!

Role of Rabi coupling, domain wall and precession of vortex molecules

(Marek Tylutki, Alessio Recati, Lev Pitaevskii and S.S,

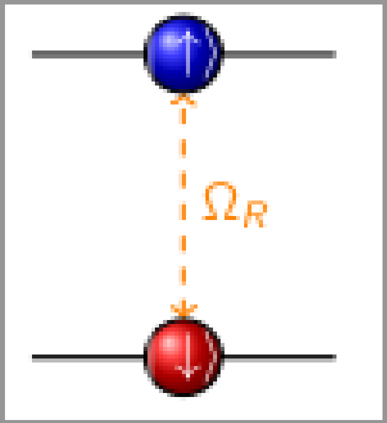
arXiv: 1601.03695)

The properties of **magnetic solitons** in coupled BEC's become even more interesting in the presence of **coherent coupling** between the two spin states (Rabi coupling)

Two GPE's with coherent (*Rabi*) coupling:

$$i\hbar\partial_t\Psi_1 = \left(H_0 + g_{11}|\Psi_1|^2 + g_{12}|\Psi_2|^2\right)\Psi_1 - \frac{1}{2}\Omega_R\Psi_2$$

$$i\hbar\partial_t\Psi_2 = \left(H_0 + g_{22}|\Psi_2|^2 + g_{12}|\Psi_1|^2\right)\Psi_2 - \frac{1}{2}\Omega_R\Psi_1$$



- Rabi coupling affects the condition of miscibility of the tro gases
- Second order phase transition due to Omega. Critical value for interspecies coupling: $\boxed{g_{12}^{cr} \equiv g + \hbar\Omega / n}$
- Conservation of total N, not of two separate numbers

Magnetic Soliton in Rabi coupled Bose-Einstein condensates

Analyticresult is obtained by

assuming uniform density ($n_1 = n_2 = n/2 = const$)

(follows from $g_1 = g_2 \equiv g$ and $\delta g << g$)

Relevant terms in Gross-Pitaveskii energy density

depend only on the phase of the two order parameters

$$E = const + \int d\vec{r}[\frac{\hbar^2 n}{4m}\left(\left|\nabla\varphi_1\right|^2 + \left|\nabla\varphi_2\right|^2\right) - \hbar\Omega_R n\cos\left(\varphi_2 - \varphi_1\right)]$$

Energy dependence on relative phase corresponds

to loss of corresponding gauge symmetry:

total number of atoms is conserved;

relative number of atoms in two spin states is not conserved

$$\boxed{\frac{\partial(N_1 - N_2)}{\partial t} = \Omega_R \int \sqrt{n_1 n_2}\sin\left(\varphi_2 - \varphi_1\right)d^3x}$$

Condition of stationarity for total energy with respect to variation of the phase yields sine - Gordon like equation (Son and Stephanov, 2002)

$$\boxed{\frac{\hbar^2}{2m}\frac{d^2}{dz^2}\varphi_1 = -\frac{\hbar^2}{2m}\frac{d^2}{dz^2}\varphi_1 = -\hbar\Omega\sin(\varphi_2 - \varphi_1)}$$

Minimum energy (ground state) takes place at $\phi_2 = \phi_1 + \mathrm{k}2\pi$

Excited state (soliton) corresponds to domain wall

$$\varphi_1(z) = -\varphi_2(z) = \arctan e^{\kappa z}$$

$$\kappa^2 = 2m\Omega_R / \hbar$$

Excitation energy fixed by surface tension of domain wall

$$\sigma = 2^{3/2}\frac{n\hbar^{3/2}}{m^{1/2}}\sqrt{\Omega_R}$$

Width of domain wall : $\kappa^{-1} = \sqrt{\hbar/(2\mathrm{m}\Omega_\mathrm{R})}$

Energy of soliton can be calculated in the presence of moving wall (Tylutki et al., 2015)

In the limit $\Omega << \delta gn/\hbar$ equation for relative phase is given by simple generalization of static Son - Stephanov eqation

Introducing dimensionless velocity $U = \dfrac{V}{\sqrt{\delta gn/m}}$

equation for the phase becomes (sine - Gordon equation)

$$(1-U^2)\partial_z^2\phi_1 + \frac{m\Omega}{\hbar}\sin(\phi_1-\phi_2) = 0 \quad \text{with} \quad U = V/\sqrt{\delta gn/m}$$

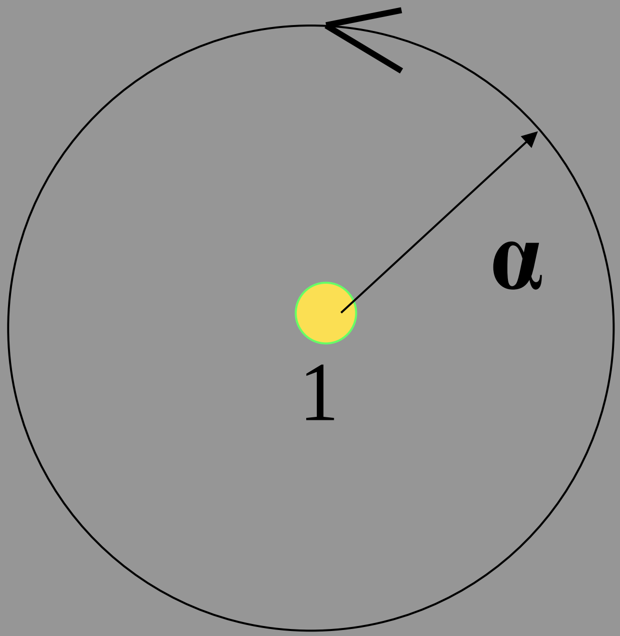
yielding

$$\boxed{\sigma(V) = \sigma(0)\frac{1}{\sqrt{1-U^2}} \approx \sigma(0) + \frac{1}{2}m^* V^2}$$

- Effective mass $m^* = m\sigma/\delta gn$ is positive (difference with usual solitons)
- Positiveness of effective mass ensures stability of domain wall against snake oscillations
- Width of domain wall decreases with velocity: $\xi(V) = \sqrt{\hbar(1-U^2)/m\Omega}$

Domain wall plays crucial role in the physics of quantized vortices in Rabi coupled BECs

Consider vortex in condensate 1

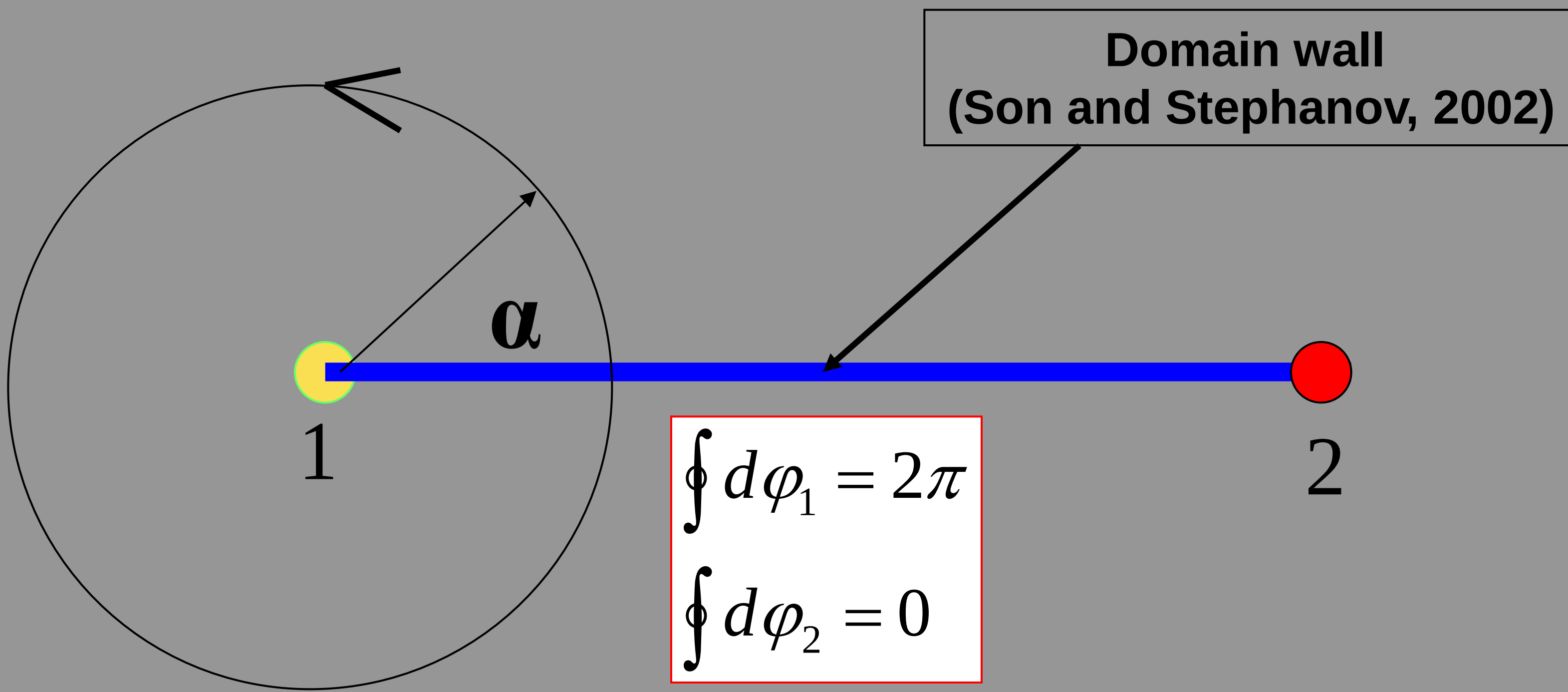


$$\oint d\varphi_1 = 2\pi$$

$$\oint d\varphi_2 = 0$$

Incompatible with blocking of relative phase
($\sin(\varphi_1 - \varphi_2) = 0$) imposed by Rabi coupling.
It results in non trivial topological scenario (domain wall)

Domain wall connects vortex 1 with vortex 2

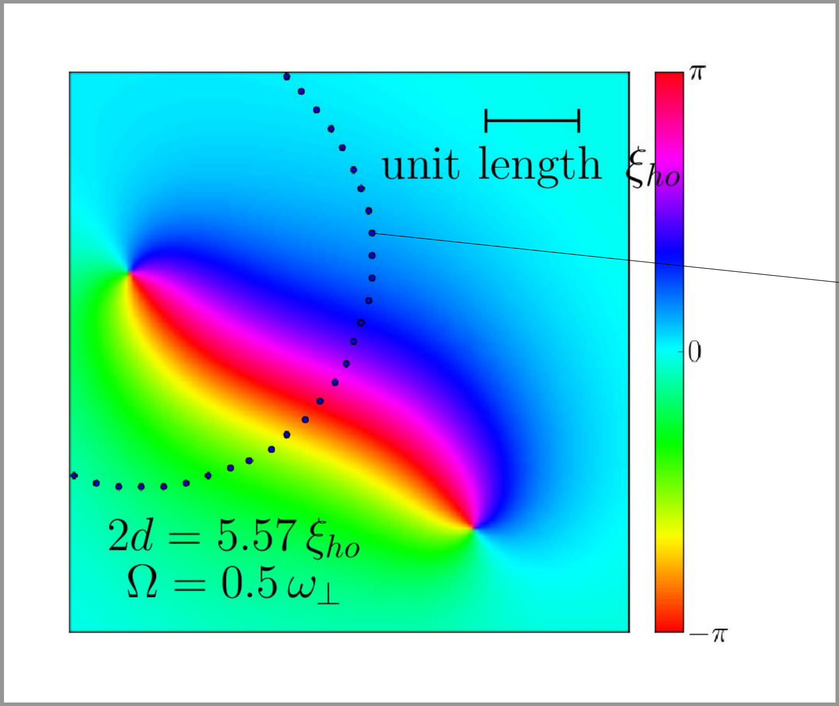


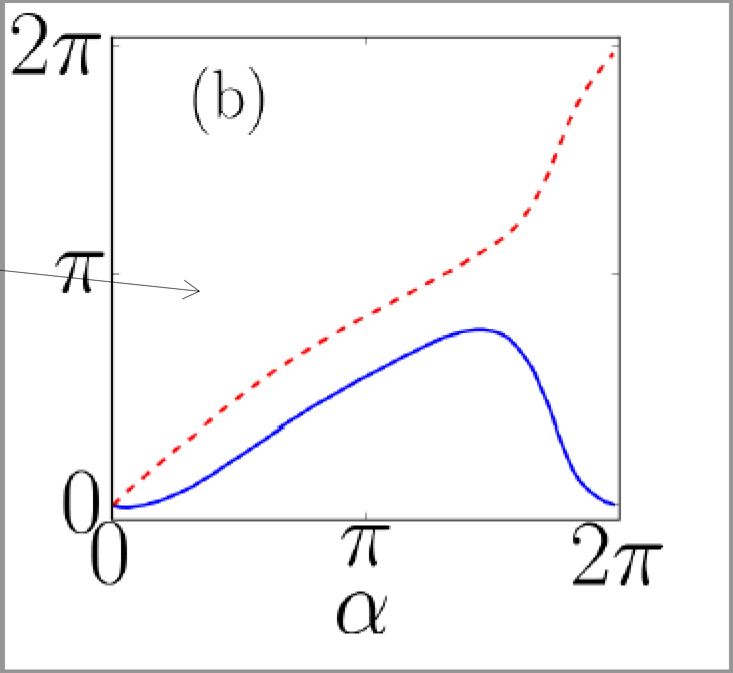
$$\oint d\varphi_1 = 2\pi$$

$$\oint d\varphi_2 = 0$$

$\sin(\varphi_1 - \varphi_2)$ vanishes everywhere except near the domain wall

Gross-Pitaevskii solution of two quantized vortices connected by a domain wall in the presence of harmonic trapping (Tylutki et al. 2015)





Vortex pairs connected by domain wall exhibit **precession**

Precession in the presence of harmonic trapping can be calculated solving TDGP equations

(first GP calculation of vortex molecules: Tsubota et al. PRA 2002; recent TDGP calculations of vortex dynamics: Nitta et al. 2012-15)

Precession can be described by macroscopic model (holds if $d >> \kappa^{-1}$)

$$E(d) = 2E_v(d) + E_{wall}(d) + E_{\text{int}}(d)$$

$$E_v(d) = E_v(1 - d^2 / R^2)$$

$$E_{wall}(d) = \int \sigma \propto \sqrt{\Omega_{Rabi}} d$$

$E_{\text{int}}(d)$ is interaction between vortices $(g_{12} \neq 0)$

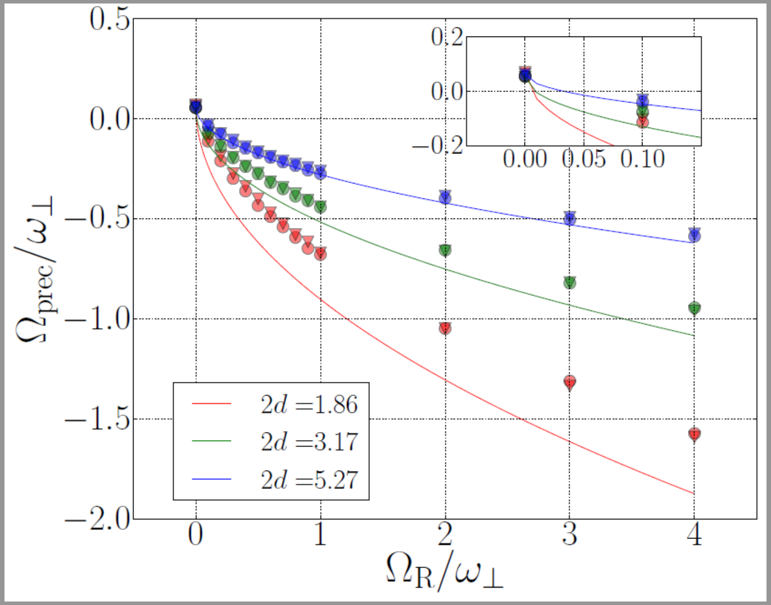
Precession frequency is given by

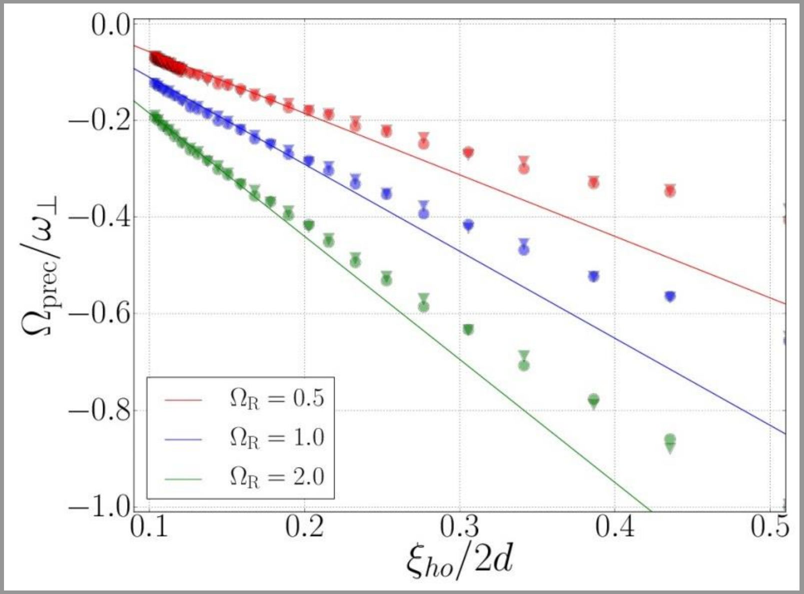
$$\boxed{\Omega_{\text{prec}} = \frac{\partial E}{\partial L_z} = \frac{\partial E}{\partial d}\frac{\partial d}{\partial L_z}}$$

with $L_z = \hbar N(1 - d^2 / R^2)^2$ and R = Thomas - Fermi radius

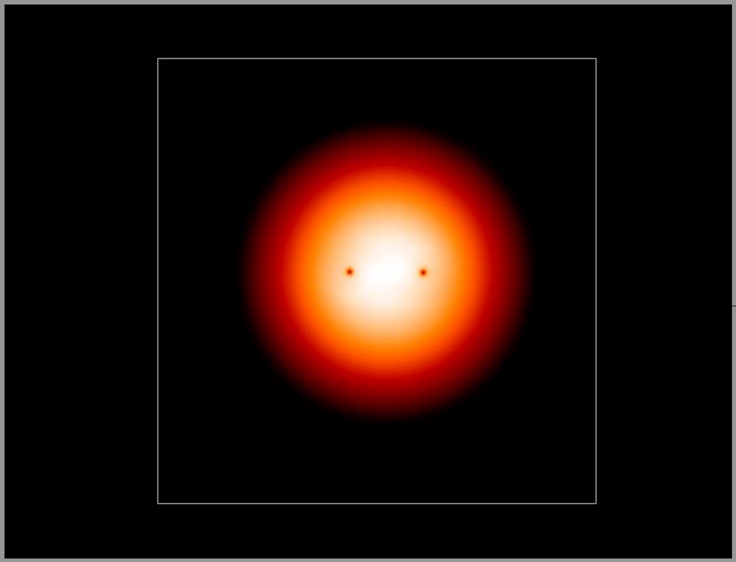
Equilibrium $(\frac{\partial E}{\partial d} = 0) \Rightarrow \Omega_{\text{prec}} = 0$

Comparison between macroscopic model and GP results. Agreement is good to the extent that the the width of the wall is small compared to its size (distance betweenn the two vortices)

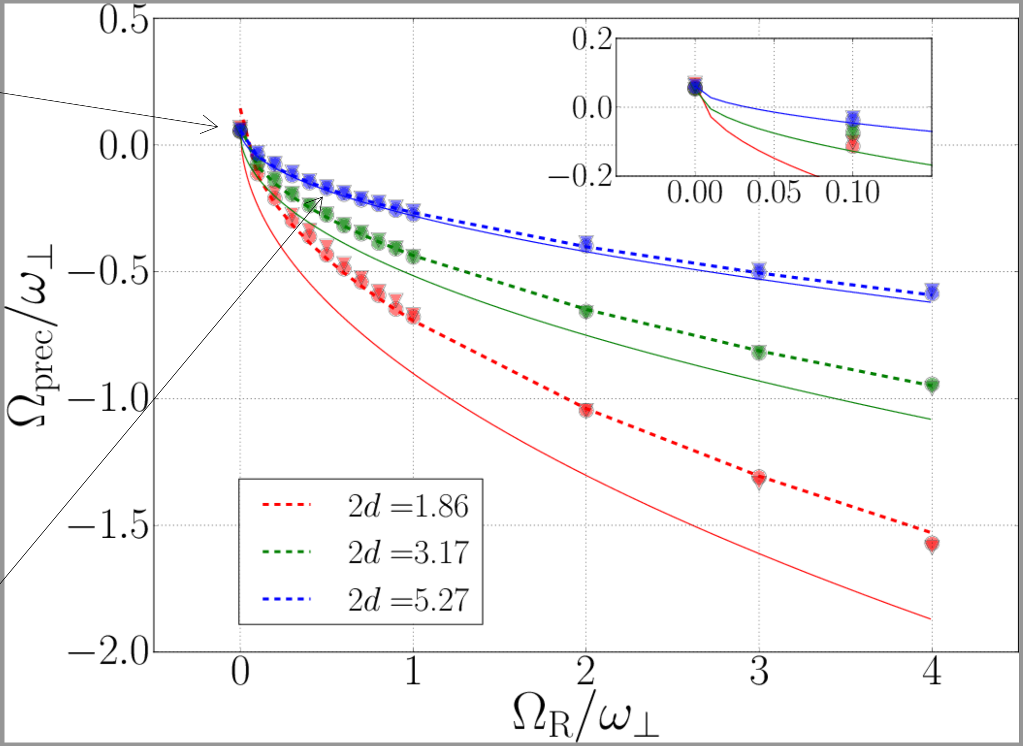


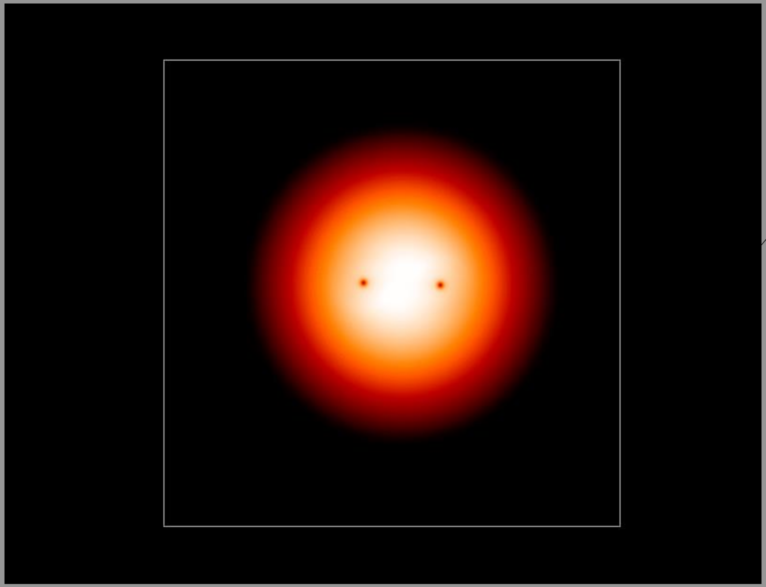


Precession of the vortex pair



If Rabi coupling is large the vortex pair rotates in opposite





MT *et al.*, *arXiv:1601.03695*

Decay of the domain wall

✔Critical value of Ω_R

$$\hbar\Omega_c = \frac{1}{3} n\delta g$$

✔ domain wall is unstable for $\Omega_R > \Omega_C$

✔"string breaking" even for $\Omega_R < \Omega_C$

✔Phonons are generated during the decay

✔ analogy with string breaking in QCD

Marek Tylutki *et al.*, *arXiv:1601.03695*

If the domain wall is too long (and in the presence of interspecies interaction !) it will exhibit fragmentation after a while with the formation of smaller domain walls, more vortices and phonon excitations (analogy with string breaking in QCD)

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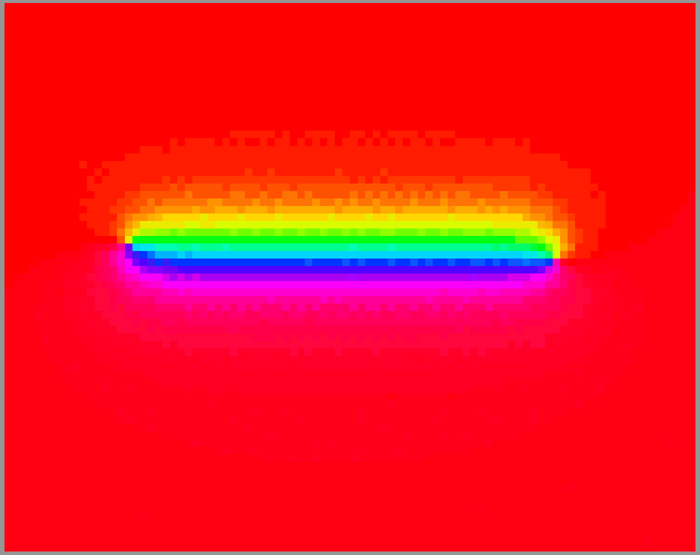
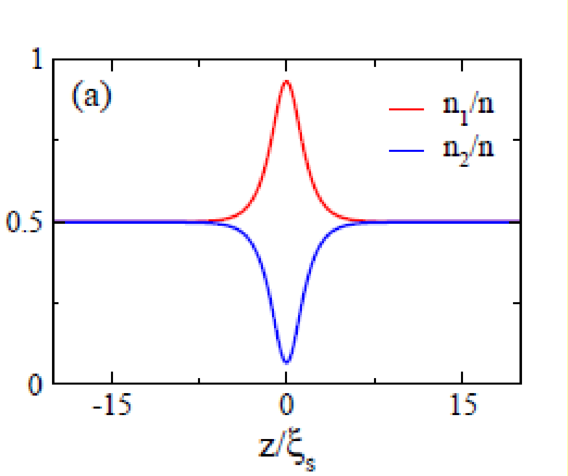


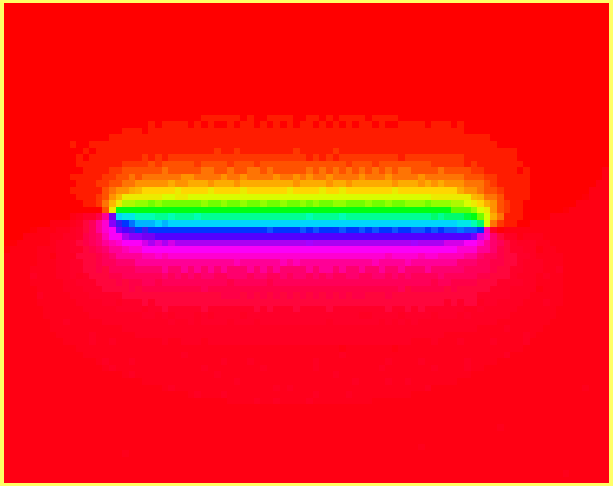
MT *et al.*, *arXiv:1601.03695*

Conclusions

- Derived analytic solution for magnetic soliton in two interacting BECs. Role of the spin sound velocity and spin healing length

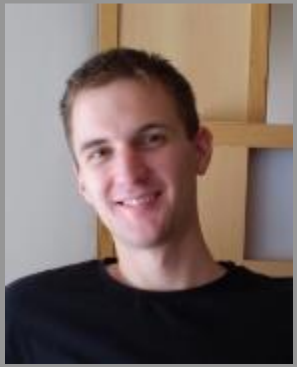


- Solitonic domain wall and precession of vortex molecules in coherently coupled BECc. Fragmentation of domain wall and analogy with QCD



Collaborators

Marek Tylutki



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