On the bound states of Schrödinger operators with δ-interactions on conical surfaces

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basque center for applied mathematics

EXCELENCIA
SEVERO
OCHOA

Problem

Let $d \geq 3$ and $\theta \in (0, \pi/2)$. We define $\mathcal{C}_{d,\theta}$, the cone with "circular" cross-section by:

$$\mathcal{C}_{d,\theta} := \{(x_1, \dots, x_d) \in \mathbb{R}^d : x_d = \cot\theta \sqrt{\sum_{j=1}^{d-1} x_j^2}\}$$

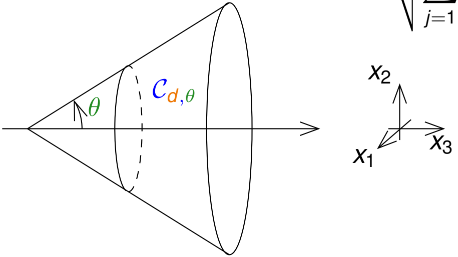

Figure: The cone $\mathcal{C}_{d,\theta}$ in dimension $d = 3$.

We are interested in the self-adjoint operator $H_{\alpha,\mathcal{C}_{d,\theta}}$ acting on $L^2(\mathbb{R}^d)$ which formally writes:

$$H_{\alpha,\mathcal{C}_{d,\theta}} = -\Delta - \alpha\delta(x - \mathcal{C}_{d,\theta}), \quad \alpha > 0.$$

Goals

Theorem [Behrndt, Exner, Lotoreichik (14)]

In dimension $d = 3$, we have:

i) $\sigma_{\text{ess}}(H_{\alpha,\mathcal{C}_{d,\theta}}) = [-\alpha^2/4, +\infty)$,

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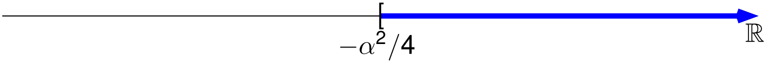

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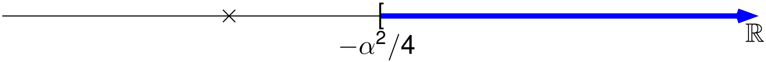

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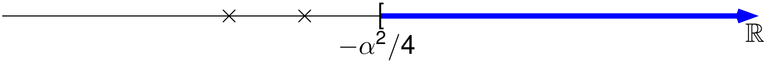

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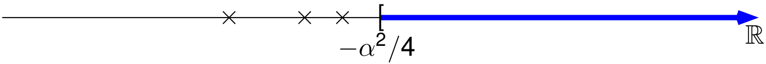

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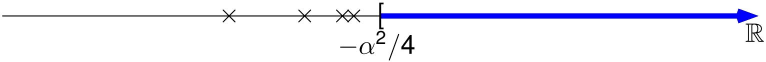

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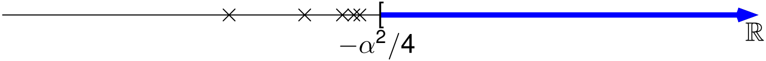

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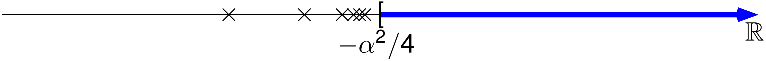

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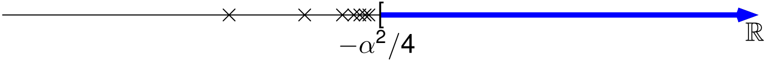

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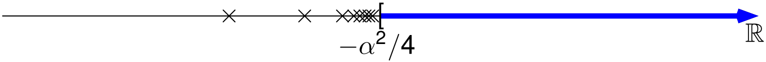

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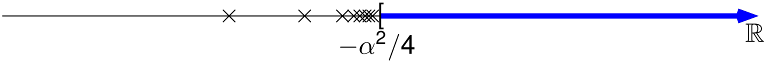

For $E > 0$, we define the counting function:

$$\mathcal{N}_{-\alpha^2/4-E}(H_{\alpha,\mathcal{C}_{d,\theta}}) = \#\{\lambda \in \sigma_{\text{dis}}(H_{\alpha,\mathcal{C}_{d,\theta}}) : \lambda < -\alpha^2/4 - E\}$$

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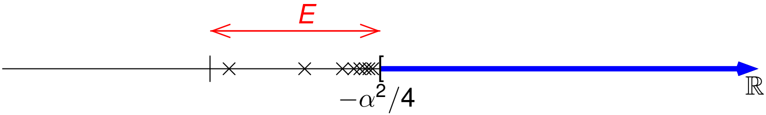

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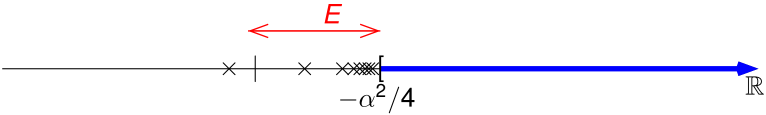

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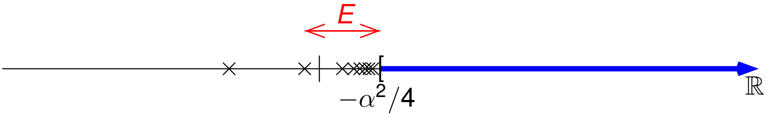

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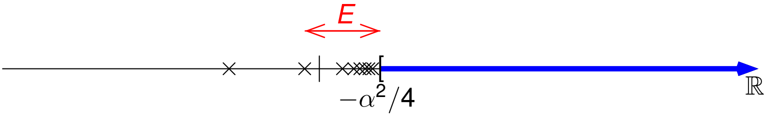

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Goals:

- For $d = 3$: behaviour of $\mathcal{N}_{-\alpha^2/4-E}(H_{\alpha,\mathcal{C}_{d,\theta}})$ when $E \to 0$.
- Structure of the spectrum in $d \geq 4$.

Laplacians and conical structures

Conical Layers:

P. EXNER, M. TATER
Spectrum of Dirichlet Laplacian in a conical layer. J. Phys. A (2010)

M. DAUGE, T. O.-B., N. RAYMOND
Spectral asymptotics of the Dirichlet Laplacian in a conical layer. Comm. Pure and Applied Ana. (2015)

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Magnetic Laplacian:

V. Bonaillie-Noël, M. Dauge, N. Popoff, N. Raymond
Magnetic Laplacian in sharp three-dimensional cones. Operator Theory Advances and Application (Birkhäuser): Proceedings of the Conference Spectral Theory and Mathematical Physics, Santiago 2014

Definition of the δ-interaction

Let $d \geq 3$, $\alpha > 0$ and $\theta \in (0, \pi/2)$. We define the quadratic form

$$Q_{\alpha,\mathcal{C}_{d,\theta}}[u] = \|\nabla u\|^2_{L^2(\mathbb{R}^d)} - \alpha \|u\|^2_{L^2(\mathcal{C}_{d,\theta})}, \quad \mathrm{dom}(Q_{\alpha,\mathcal{C}_{d,\theta}}) = H^1(\mathbb{R}^d).$$

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Proposition [Behrndt, Exner, Lotoreichik (14)]

The quadratic form $Q_{\alpha,\mathcal{C}_{d,\theta}}$ is closed and semi-bounded on $L^2(\mathbb{R}^d)$. Therefore, we denote $\mathsf{H}_{\alpha,\mathcal{C}_{d,\theta}}$ the associated self-adjoint operator given by its Friedrichs extension.

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Reduction to $\alpha = 1$ Let $u \in \mathrm{dom}(Q_{\alpha,\mathcal{C}_{d,\theta}})$, we define $\hat{x} = \alpha^{-1}x$. As $\mathcal{C}_{d,\theta}$ is dilatation invariant we get:

$$\frac{Q_{\alpha,\mathcal{C}_{d,\theta}}[u]}{\|u\|^2_{L^2(\mathbb{R}^d)}} = \alpha^2 \frac{Q_{1,\mathcal{C}_{d,\theta}}[\hat{u}]}{\|\hat{u}\|^2_{L^2(\mathbb{R}^d)}}.$$

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From now on, we drop the index 1: $Q_{1,\mathcal{C}_{d,\theta}} = Q_{\mathcal{C}_{d,\theta}}$ and $\mathsf{H}_{1,\mathcal{C}_{d,\theta}} = \mathsf{H}_{\mathcal{C}_{d,\theta}}$.

Main result

Theorem [Lotoreichik, O.-B. (15)]

Let $\theta \in (0, \pi/2)$.

i) In dimension $d \geq 3$, $\sigma_{\text{ess}}(\mathsf{H}_{\mathcal{C}_{d,\theta}}) = [-1/4, +\infty)$.

ii) In dimension $d = 3$, we have

$$\mathcal{N}_{-1/4-E}(\mathsf{H}_{\mathcal{C}_{d,\theta}}) \sim \frac{\cot\theta}{4\pi}|\ln E|, \quad E \to 0.$$

iii) In dimension $d \geq 4$, $\sigma_{\text{dis}}(\mathsf{H}_{\mathcal{C}_{d,\theta}}) = \emptyset$.

(hyper-)cylindrical coordinates

Let $(r, z, \phi) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{S}^{d-2}$ be the cylindrical coordinates, for all $k \in \{1, \ldots, d-2\}$:

$$x_k = r \left(\prod_{p=1}^{k-1} \sin \phi_p \right) \cos \phi_k, \quad x_{d-1} = r \prod_{p=1}^{d-2} \sin \phi_p, \quad x_d = z.$$

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$\mathbb{R}^d$ becomes $\mathbb{R}_+^2 \times \mathbb{S}^{d-2}$. $\mathcal{C}_{d,\theta}$ becomes $\Gamma_\theta \times \mathbb{S}^{d-2}$:

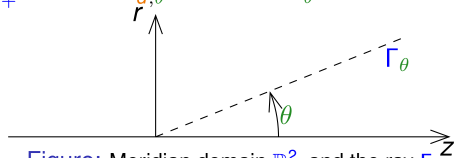

Figure: Meridian domain $\mathbb{R}_+^2$ and the ray Γ_θ

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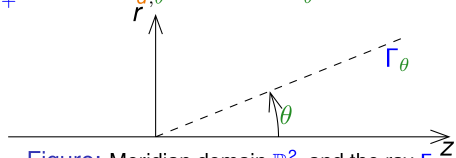

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The quadratic form $Q_{\mathcal{C}_{d,\theta}}$ is expressed as

$$Q_{\mathcal{C}_{d,\theta}}[u] = \int_{\mathbb{R}^2_+ \times \mathbb{S}^{d-2}} (|\partial_r u|^2 + |\partial_z u|^2 + r^{-2} \|\nabla_{\mathbb{S}^{d-2}} u\|^2) r^{d-2} \mathrm{d}r \mathrm{d}z \mathrm{d}\mathfrak{m}_{d-2}(\phi)$$
$$- \int_{\Gamma_\theta \times \mathbb{S}^{d-2}} |u(s, \phi)|^2 \mathrm{d}\gamma_\theta(s) \mathrm{d}\mathfrak{m}_{d-2}(\phi).$$

Fiber decomposition

Decomposing into spherical harmonics, we get the family of quadratic forms:

$$Q_{\Gamma_\theta}^{[l]}[u] = \int_{\mathbb{R}_+^2} \left(|\partial_r u|^2 + |\partial_z u|^2 + \frac{l(l+d-3)}{r^2}|u|^2 \right) r^{d-2} \mathrm{d}r\mathrm{d}z$$
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The quadratic forms do not depend on k and their domains are:

$$\mathrm{dom}(Q_{\Gamma_\theta}^{[l]}) = \begin{cases} \{u : u, \partial_r u, \partial_z u \in L^2(\mathbb{R}_+^2, r^{d-2}\mathrm{d}r\mathrm{d}z)\}, & l = 0, \\ \{u : u, \partial_r u, \partial_z u, r^{-1}u \in L^2(\mathbb{R}_+^2, r^{d-2}\mathrm{d}r\mathrm{d}z)\}, & l > 0. \end{cases}$$

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Remark:

If $(l, d) = (l, 3)$ and $l > 0$ then for all $u \in \mathrm{dom}(Q^{[l]}_{\Gamma_\theta})$, $u(0, z) = 0$.

Flat metric

Proposition [Lotoreichik, O.-B. (15)]

Let $d \geq 3$ and $l \in \mathbb{N}$ such that $(d, l) \neq (3, 0)$. Then $Q_{\Gamma_\theta}^{[l]}$ is unitarily equivalent to the quadratic form

$$\int_{\mathbb{R}_+^2} |\partial_r \tilde{u}|^2 + |\partial_z \tilde{u}|^2 + \frac{\gamma(d,l)}{r^2} |\tilde{u}|^2 \mathrm{d}r\mathrm{d}z - \int_{\mathbb{R}_+} |\tilde{u}(s \sin\theta, s\cos\theta)|^2 \mathrm{d}s,$$

with $\gamma(d,l) = l(l+d-3) + (1/4)(d-2)(d-4)$ and $\tilde{u} \in H_0^1(\mathbb{R}_+^2)$,

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with $\gamma(d, l) = l(l + d - 3) + (1/4)(d - 2)(d - 4)$ and $\tilde{u} \in H^1_0(\mathbb{R}^2_+)$,

Proof: Let $(d, l) \neq (3, 0)$: For $u \in \mathrm{dom}(Q^{[l]}_{\Gamma_\theta})$, we let $\tilde{u} = r^{(d-2)/2} u$.

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with $\gamma(d,l) = l(l+d-3) + (1/4)(d-2)(d-4)$ and $\tilde{u} \in H_0^1(\mathbb{R}_+^2)$,

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$$\int_{\mathbb{R}_+} |\partial_r (r^{-(d-2)/2}\tilde{u})|^2 r^{d-2} \mathrm{d}r = \int_{\mathbb{R}_+} |\partial_r \tilde{u}|^2 \mathrm{d}r + \int_{\mathbb{R}_+} \frac{\tilde{\gamma}(d)}{4r^2}|\tilde{u}|^2 \mathrm{d}r + \frac{d-2}{2}\lim_{r\to 0}(r^{(d-3)}|u|^2).$$

Flat metric

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Reduction to $(d, l) = (3, 0)$

Proposition [Lotoreichik,O.-B. (15)]

Let $d \geq 3$ and $l \in \mathbb{N}^*$. $Q^{[l]}_{\Gamma_\theta}$ can generate discrete spectrum only if $(d, l) = (3, 0)$.

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Thanks to the min-max principle: $\inf \sigma(Q_{\Gamma_\theta}^{[l]}) \geq -(1/4)$.

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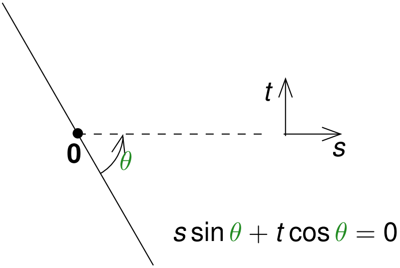
Consequence: We focus only on $(d, l) = (3, 0)$ to prove the accumulation of the eigenvalues.

Asymptotics of the counting function



In these variables the quadratic form reads:

$$Q_{\Omega_\theta}[u] = \int_{\Omega_\theta} (|\partial_s u|^2 + |\partial_t u|^2)(s\sin\theta + t\cos\theta)\,ds\,dt - \int_{s>0} |u(s,0)|^2 s\sin\theta\,ds$$

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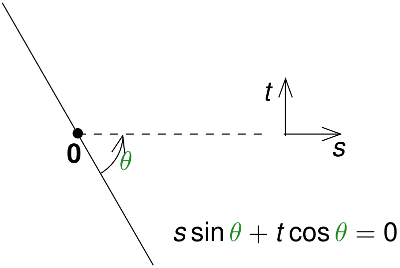

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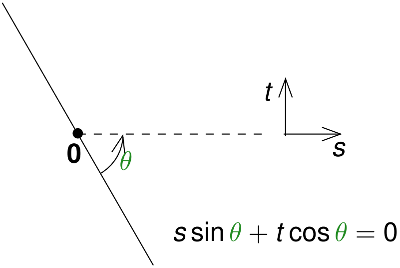

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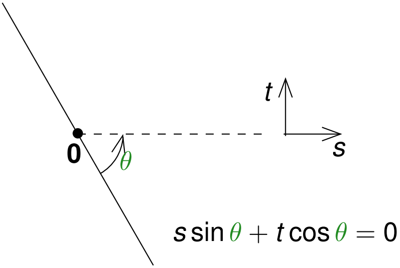
Now, we bound Q_{Ω_θ} by two quadratic forms using Dirichlet and Neumann bracketing:

$$Q^{\mathrm{N}}_{B(E)} \le Q_{\Omega_\theta} \le Q^{\mathrm{D}}_{\mathrm{Hst}(E)}$$

Where, $Q^{\mathrm{N}}_{B(E)}$ and $Q^{\mathrm{D}}_{Hst(E)}$ are tensored quadratic forms.

Asymptotics of the counting function



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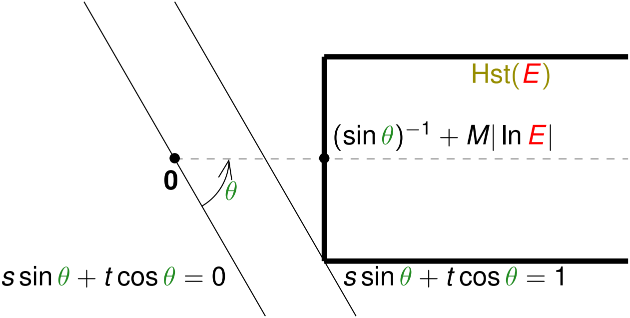
Where, $Q^{\mathrm{N}}_{B(E)}$ and $Q^{\mathrm{D}}_{Hst(E)}$ are tensored quadratic forms. It yields

$$\mathcal{N}_{-1/4-E}(Q^{\mathrm{D}}_{\mathrm{Hst}(E)}) \leq \mathcal{N}_{-1/4-E}(Q_{\Omega_\theta}) \leq \mathcal{N}_{-1/4-E}(Q^{\mathrm{N}}_{B(E)})$$

Lower bound on the counting function



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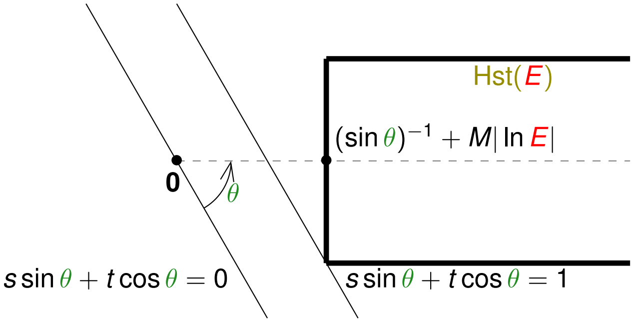


For $u \in \mathrm{dom}(Q_{\Omega_\theta})$ such that $u = 0$ on $\Omega_\theta \setminus \overline{\mathrm{Hst}(E)}$ we define

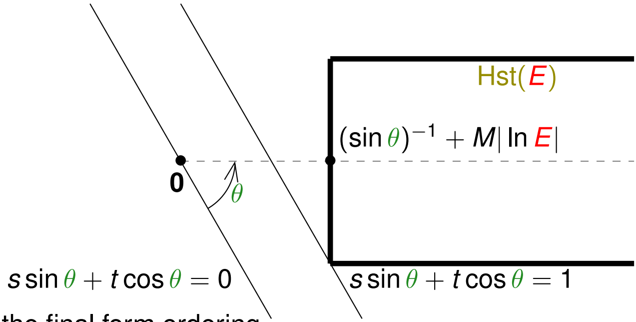
$$\tilde{Q}^{\mathrm{D}}_{\mathrm{Hst(E)}}[u] = Q_{\Omega_\theta}[u].$$

We get the form ordering:

$$Q_{\Omega_\theta} \leq \tilde{Q}^{\mathrm{D}}_{\mathrm{Hst(E)}} \equiv \hat{Q}^{\mathrm{D}}_{\mathrm{Hst(E)}},$$

where $\hat{Q}^{\mathrm{D}}_{\mathrm{Hst(E)}}$ is the expression of $\tilde{Q}^{\mathrm{D}}_{\mathrm{Hst(E)}}$ in the flat metric.

Lower bound on the counting function



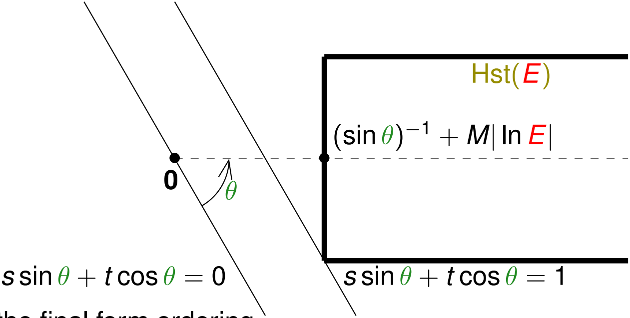
We get the final form ordering

$$Q_{\Omega_\theta} \leq \hat{Q}^{D}_{\mathrm{Hst}(E)} \leq Q^{D}_{\mathrm{Hst}(E)},$$

where $Q^{D}_{\mathrm{Hst}(E)}$ quadratic form of a tensored operator on $L^2(\mathrm{Hst}(E))$ of the shape:

$$-\partial_t^2 - \delta_{t=0} - \partial_s^2 - \frac{1}{4s^2\sin\theta}$$

Lower bound on the counting function



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$$\underbrace{-\partial_t^2 - \delta_{t=0}}_{\lambda_1(E)>1/4} - \partial_s^2 - \frac{1}{4s^2 \sin\theta}$$

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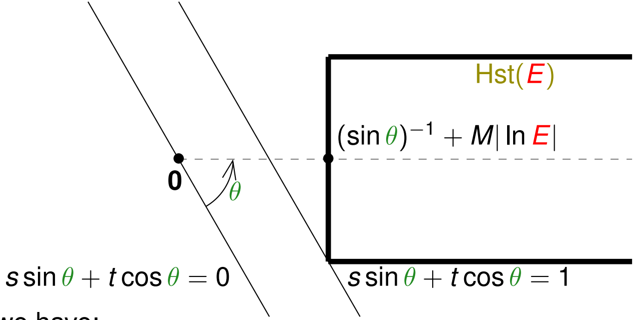

Finally we have:

$$\mathcal{N}_{-1/4-E-\lambda_1(E)}\left(-\partial_s^2 - \frac{1}{4s^2 \sin\theta}\right) \leq \mathcal{N}_{-1/4-E}(Q_{\Omega_\theta})$$

We choose $M > 0$ such that $1/4 + E + \lambda_1(E) = \mathcal{O}(E|\ln E|)$.

P. Exner, K. Yoshitomi
Asymptotics of eigenvalues of the Schrödinger operator with strong δ-interaction on a loop. J. Geom. Phys. (2002)

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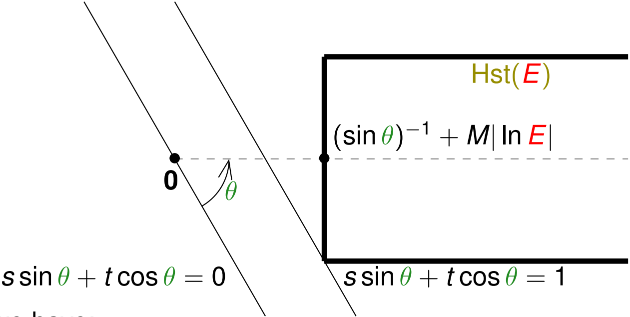

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W. KIRSCH, B. SIMON
Corrections to the classical behavior of the number of bound states of Schrödinger operators. Ann. Phys. (1988)

Eskerrik asko zure arretagatik !

Thank you for your attention !